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The Teaching of
Arithmetic
AND
Elementary Mathematics

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BASIL BLACKWELL, OXFORD

1938

Printed in Great Britain for BASIL BLACKWELL & MOFF LTD.
by the KEMP HALL PRESS LTD in the City of Oxford

PREFACE

THIS book has been written with several aims in view:

1. To give a short survey of the teaching of Arithmetic and Elementary Mathematics, its problems and possibilities, with special reference to work in Senior, Modern or Central Schools. Stress has been laid on the use of Mathematics in our daily life and in the world around us.

2. To provide a Teachers' guide to the 'New Freedom' series of Arithmetic and Elementary Mathematics Books.

3. To offer suggestions in practical work, the history of Mathematics and the human aspects of the subject, which could be used to invigorate the normal Secondary School course in its earlier stages. When properly taught, Mathematics makes a universal appeal and is as fascinating as popular science.

4. To give an account of Mathematics teaching for those whose main interest does not lie in the subject. Many teachers who are not specially qualified in the teaching of Arithmetic are asked to teach this subject, and the majority of students in training-colleges are expected to have some knowledge of it.

No attempt has been made to give a systematic account of Mathematics teaching in Secondary and similar schools, for the excellent publications of the Mathematical Association amply fulfil the needs of teachers at this stage.

In conclusion, I wish to acknowledge my indebtedness to the Controller of H.M. Stationery Office for permission to quote from various Board of Education publications. I should like to thank Professor H. T. H. Piaggio, D.Sc., M.A., for some suggestions; and my sister, Miss Grace Sumner, for checking the proofs.

W.L.S.

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CHAPTER I

WHY DO WE TEACH MATHEMATICS IN SCHOOLS?

IT has been said of Arithmetic that it is the least popular of the school subjects. In many quarters this attitude has been accepted almost as a tradition and is sometimes fostered by the outlook of the teacher rather than by that of the children. The bitterest critics of mathematical studies in schools are willing to admit the absolute necessity of a knowledge of the fundamentals of arithmetic and mensuration, if only from the utilitarian point of view. We live in a measured world, and, moreover, in one where there is an ever-increasing search towards greater control by the application of the yardstick. The family budget has to be balanced, rates and taxes paid, and what is more important from the point of view of good citizenship, spent wisely. Quantities of every type of commodity have to be weighed or measured and various prices fixed. The very foundations of the world's complex commercial systems depend on this. Time has to be noted accurately, for many things in ordinary life, in travelling and in science depend on it. Arithmetic has got beyond the stage of commercial necessity and the assessment of wealth; it is the handmaid of practical sciences which are as far removed as engineering and medicine. Our safety when we walk over bridges or travel in modern vehicles depends on it; and even the diagnosis of disease is often the result of mathematical principles applied to medical knowledge. Throughout this book we shall stress the value of mathematics in helping us to understand the world and life around us. History, geography, economics, navigation, many things in our daily routine and some of the finest achievements of the mind of man and what led up to them, are only properly compre-

hended by the application of mathematical principles. In short, elementary mathematics properly esteemed, is not only a necessary part of the equipment of the man interested primarily in monetary affairs, but it is essential for an understanding of the further obligations and privileges of intelligent citizenship.

As the advantages given to our everyday life by elementary mathematics are on the whole of much greater antiquity than those of science, we are sometimes inclined to overlook them. Calculations involving enormous magnitudes have been made accessible, even to those of moderate intelligence, by our positional numeration; the use of symbolism and formulae in algebra has made quite complex relations both simple to grasp and easy of manipulation; and the graph (that invention of Descartes which was partly anticipated by the Greeks) enables us to see at a glance in pictorial form, the law connecting dependent events, or changes during the course of time.

In former years, the chief aim in teaching simple mathematics was to stress its value as 'mental' training. This attitude is summed up by saying that:

(a) Arithmetic demands the power to reason and to concentrate.

(b) It offers an inexhaustible field of simple exercises in logic.

(c) It favours the suppression of 'superfluous movements and tension'.

(d) Through arithmetic calmness, confidence and 'uniform emotional disposition' may be obtained.

(e) It develops the use of imagery.

There is, of course, a large measure of truth in all this, but it is also true of many subjects when they are correctly taught; and indeed most of the desirable qualities mentioned in the passage quoted above could be equally fostered by a course in eurhythmics! Utility and culture are by no means

incompatible, and the old ideas about cultural school subjects have undergone considerable modifications in recent years. Cultural values are enhanced when the subject is made practical and useful in the lives of men, and when the human element is introduced. There have been not a few poets and other writers who have been repelled by 'cold, inhuman calculators'. When properly realized, mathematics is no longer cold and austere, and it is far from being deficient in human values.

The alleged dullness of mathematical studies is a myth, the product of a century and a half or more of mechanical methods and teaching; it is not inherent in the subject and need no longer be true of it to-day. Sometimes considerable harm has been done to the cause of arithmetic by giving the early stages to teachers who have no real love for, or 'faith in the subject'. The attitude of 'This is done as a necessity and as a mental discipline' often prejudiced the child from the start. A small percentage of children like doing arithmetic in the same way that in later years they would like a game of chess; it 'works out' and there is satisfaction in getting the right answer. It is a 'clean subject' requiring little writing, and if the truth be told, little real thinking after a few cardinal rules or tricks have been mastered. This is a bad conception of arithmetic and reflects wrong traditions. Professor L. Hogben in his *Mathematics for the Million* has written about the subject from the point of view of its position and use in our lives, both as individuals and as members of society; and with special reference to the history of the subject. The book was read with avidity by hundreds of thousands of adults in all walks of life. May we not suppose that in their earlier days many of these readers, while still at school, asked for mathematical bread and were given a stone? The obvious inference is that many people could not have grasped the real nature of mathematics when they were at school, but when the subject was presented to them

in a different light at a later date they welcomed it with enthusiasm.

We, who have lived in the twentieth century, have watched modern science grow up and seen many inventions made in response to man's ever-growing needs. In a parallel way, researches in mathematics were made many centuries ago to satisfy the demands of men, in the matter of assessing and increasing commercial wealth, measuring land and time, and finding latitude and longitude when travelling. Arithmetic is almost a consequence of the acquisitive instinct, for whenever man had personal or family possessions he sought for some means of measuring or numbering them. 'Mathematics', says Hogben, 'is the mirror of civilization'. Even many of the more academic branches of mathematics found their origin in man's needs, whether they concerned his possessions, his health, his daily routine or even his religion. Wherever possible, mathematics should be approached from its historical aspect. There has been a tendency to devitalize elementary mathematics as the centuries have passed. It has been taken from its historical context, and been stripped of its human and practical values. It has been frequently 'polished', placed in an artificial setting, and has reappeared in nineteenth and early twentieth-century text-books looking very uneasy and ornamental. Mathematics, despite the antiquity of some of the most elegant portions of the subject, is not a static thing. It is progressive and vital, and it is in this light that it must be taught. Mathematical laws are *not* the 'Laws of God', transcending man's mind, and giving a system which shows the real pattern of the Universe. They are man-made, largely limited by human intuition, and as such they bear all the qualities of human activity. In teaching mathematics, care should be taken to avoid giving the impression that there is something eternally and objectively absolute about the subject. As the great arithmetician Kronecker said: 'God made

the integers, and man has had to do the rest'; and even the first part of the statement needs careful qualification. It is better to present mathematics as a connected series of tools than as a statement of eternal values.

Mathematical statements are couched in a special language and to understand the subject we must know something about that language. Many children and even adults find mathematical symbolism difficult or repellent. Here a little history of mathematics would reveal that our present notation is very neat and compact, and stands in marked contrast to the difficult and clumsy methods which were used at the beginnings of various parts of the subject. Even a simple quadratic equation appeared to be of appalling difficulty and complexity when set down in ancient form. Frequently in daily life, simple arithmetical or mathematical statements, with every parade of innocence, have been used deliberately to hoodwink the unwary. Arithmetic should teach the student to look behind the figures, to endeavour to find their meaning and to ask whether it is valid and important. The milliner's 1s. 11 $\frac{3}{4}$ d. is practically 2s., but at first most people count upwards from the shilling; 'just fourpence a day' means an expenditure of over £6 a year; and to fill in the results of only 12 (football) matches means that there are 531,441 different ways of doing this, only one of which is correct. We must not be over-awed by figures; neither must we be deceived by them, even though they are correct when accepted at their face value.

Our system of numeration is so easily set down and provides such facilities for mechanical manipulation, that we tend to forget the thing behind the number, and the number behind the symbol. The number of things which most people can grasp and picture mentally is very small.

The old-fashioned mechanical arithmetic and mathematics tended to give the impression that exact (usually integral) results were the final achievements of calculation.

We have first to measure quantities before we can manipulate the numbers which represent them. Measurement can never be exact, and there are always errors and approximations. Mathematics which is to be useful either to ordinary life or to science has to concern itself with these problems. Intelligent approximation, the nature of errors, where they are likely to be found, their limits and how errors in data are likely to affect the final result after the various processes of mathematical manipulation have been performed, are of paramount importance in all stages of mathematics. We find these questions arising in the simple problems on the decimalization of money, or indeed in any question involving everyday measurements; and on the other hand, in surveying, navigation and the application of advanced mathematics to equally advanced science.

In the early part of the seventeenth century Galileo visualized the Universe as a species of mathematical machine, though in some ways he was anticipated by the Greeks. The history of science since his time has shown how more branches of the subject have become 'exact': astronomy and mechanics were followed by physics, and then came mathematical methods in chemistry. Mathematics has enabled scientists to express in convenient terms the results of their observations and subsequent thought; and it has given them the means of checking theories, extending their general principles, predicting further results, and making new discoveries of far-reaching importance. 'Wireless waves' were 'discovered' mathematically by Maxwell more than twenty years before they were actually found by Heinrich Hertz. The planets Neptune and Pluto, were 'mathematical discoveries' and were found by the telescope after arithmetical working had indicated their positions in the sky. Mathematics makes for efficiency in science as well as in daily life!

Descriptive and qualitative aspects of science are impor-

tant, but science is transformed into a much more powerful and beautiful system when it becomes quantitative and 'mathematical'. Children should be led to see the enormous difference between the qualitative and quantitative description of a thing or a process. Many cases from daily life and science will readily come into the mind, such as the making of a cake or the dispensing of a bottle of medicine, which must be done quantitatively as well as qualitatively. 'At last', said a great authority on the construction of musical instruments in 1770, 'I am able to tell you the secrets of their manufacture for I can give you all measurements and quantities'.

Biology, psychology and economics find mathematical processes such as statistics, graphs and other forms of expression of the greatest use. Insurance companies work out their premiums by statistical methods, sociologists find graphs of considerable value, and even in the useful arts 'rule of thumb' methods are giving place to accurate measurements and formulae.

To revert to a much simpler stage of the subject, arithmetic and elementary mathematics are often the only subjects with which the child meets in early years which savour of an 'exact' science. Here the aim is quite clear; and satisfactory teaching should lead to logical thought and precise statements. Every teacher should be a teacher of the mother-tongue; and careless, ambiguous statements in mathematics tend to destroy the value of the subject. There is much more in arithmetic, even in its primary stages, than the accurate working of certain standard forms. It must concern itself with clear statements of measurements, quantities and relationships, and the expression and interpretation of results. In all mathematical teaching, stress should be laid on rough estimates and methods of checking results. Systems of setting down and working problems should be employed which will allow both the argument and the mathematical

calculation to be checked at each stage. Reference to applied mathematics in engineering and navigation, where life and property have to be protected, will serve to show the great importance of independent methods of verifying an answer.

Although we have stressed the fact that mathematics is so important in our daily lives, in our science, our hobbies, indeed in everything which deals with measured quantities, it must not be entirely forgotten that the subject can be pursued for its own sake. The number of ordinary people who are attracted by this aspect of it is perhaps not very great, particularly if we are thinking of the subject from the central or senior school point of view. If there are potential Wranglers in our classes they will not have suffered by a utilitarian and historical treatment of the subject—indeed, they may benefit by the escape from ‘sterile academics’—and for the ordinary citizen a pursuit of the aims which we have delineated in this introduction will make mathematics enjoyable, useful and human.

There are particular aims which will help to determine syllabuses and treatment in different types of schools and these will be mentioned in the appropriate chapters.

CHAPTER II

VERY EARLY STAGES IN ARITHMETIC

PSYCHOLOGISTS and biologists have traced a similarity between the stages of development of the individual from his earliest years and the development of a whole race of people in the process of civilizing itself. The individual epitomizes the race. This parallelism between ontogenesis and phylogenesis is known as the theory of recapitulation, and although it may only be regarded as a broad guide, it serves to suggest many useful things in our modes of approaching various educational subjects. What a step forward it was in the history of human thought when some 'prehistoric Newton' was able to read into seven similar things the abstract idea of the number seven! Mankind must have struggled for centuries before this stage could be reached, and yet we are often inclined to hurry infants towards a similar attainment. It is trite to say that the early approach to the ideas of number is important and sometimes difficult. The importance is a greater one than the mere necessity of forming a correct foundation for future work in arithmetic. The child's later outlook on a subject is largely determined by feelings of satisfaction and happiness on the one hand, or drudgery, dullness and frustration on the other. The teacher's difficulties in the early stages are manifold. Children start their school life with various fragments of knowledge picked up at home, which vary with their environment and the intelligence of their parents and others with whom they have come into contact. In any particular child, the sensory activities are often in different stages of development; also, some

children are by nature tactiles (or tactuals) learning chiefly through the sense of touch, others visuals (learning through what is seen), and a small minority learn chiefly through their ears and are called audiles. Ideally, every child needs separate study, and different methods of approach for the particular problems which he presents. In children of normal intelligence partial inability to learn through a particular sense is compensated in some measure by the activities of the other senses. Special treatment for backward or mentally defective children in early arithmetic has been dealt with later.

With efficient and sympathetic teachers practically all children enjoy the opportunities for activity, for movement, for elucidating simple puzzles, for the use of apparatus, paper and pencil which arithmetic affords. It was commonly asserted that arithmetic should not be commenced as early as work in the mother-tongue. Some teachers would even exclude arithmetic from the Infant School. Investigations in English and American schools have shown that there is no need for this. 'Children soon find in arithmetic a field in which they can measure their growing mastery of the tool and their growing powers of concentration. The child's natural interest in, and curiosity regarding shape, size and number is the justification for his beginning arithmetic in the Infant School.'¹ Dislike of the subject usually comes later in the Primary and Post-Primary Schools and this often reflects the methods of teaching it. The main causes of failure on the part of the child are a lack of self-confidence leading to the fear of failure, which brings general dissatisfaction with the subject. In arithmetic nothing succeeds like success. Sometimes the child has been drilled so mechanically that it does not grasp the genius of the subject and what it is trying to do. The writer recently investigated the cases of a large number of school children and 'non-

¹*Infant and Nursery Schools*. H.M.S.O., 1933.

mathematical' university students who alleged that they did not like arithmetic. The main causes were: (1) Hurry in the early stages and a consequent failure to get many 'sums' right, leading to a lack of self-confidence, which in turn made them avoid the subject and seek expression in others, (2) a lack of appreciation of what arithmetic really sets out to do, and (3) a feeling that the subject was 'artificial' and had little application, as it stood, to real life. A common remark is: 'I dislike arithmetic because I could never do it'; or again, 'I could do arithmetic but I never knew what it was about'. The apparent paradox in the last statement shows that stress had been laid on mechanical drill to the exclusion of nearly everything else. From what we have just said it would appear that children should be very carefully graded and given every encouragement, particularly in the face of failure, to grasp the early ideas of number.

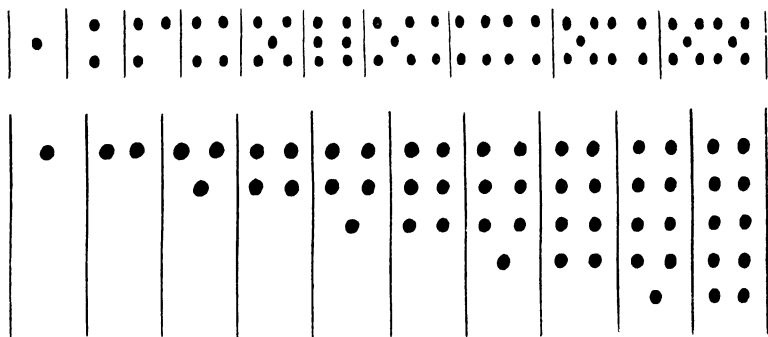
The first step is counting, and the aim here will be to enrich and consolidate home habits. Reciting or chanting nursery rhymes dealing with simple numbers in progressive order, especially if it is accompanied by appropriate bodily actions, is interesting and useful. Many examples will suggest themselves to the teacher. 'This little pig went to market'; 'Ten little nigger boys'; 'One, two, buckle my shoe'; 'Mary's at the cottage gate'; 'Green grow the rushes'; 'The twelve days of Christmas', etc. Incidentally, many of these songs have great use in teaching, because of their rhythmic and traditional significance. Early work should aim at correlating the impressions gained through ear, eye, sense of touch and muscular sense in general. The symbol which represents the number has to be connected with the articulated sound by which we recognize it. This is not enough, for it is necessary to ascertain that the child can use it intelligently in simple speech and understand it when it is written as a word. The things which have to be connected

- (1) The figure (4).
- (2) The written and spoken word (four). . .
- (3) The pattern in which the things may be arranged . . .

The shape of the figure often gives trouble, and children will talk about simple numbers which they have not grasped; or again, the order of the numbers may be developed mechanically without any understanding that, for instance, $6 + 1 = 7$. Number may be used in two ways: cardinally and ordinally. The cardinal use is the oldest and, to a certain extent, is even realized by animals and birds which are able to appreciate the number of their young. The ordinal use implies counting in correct order, and that having reached a certain number it can always be followed by another. Some of the early difficulties of infants when using numbers are due to a confusion of the ordinal and cardinal uses. Many varieties of objects must therefore be used for practice in counting. It may be found convenient to co-ordinate simultaneously the visual, tactile and muscular impressions by using the Montessori method of having large figures cut out in sandpaper and glued to smooth boards. By rubbing his finger round the figure, the child remembers the shape in a much better way than by just looking at it, or by making feeble early attempts to write it. Although it has been stressed that no opportunity must be lost in early work of using the touch, muscular and rhythmic senses, it must not be forgotten that this applies equally well to fields of educational activity which come much later. Imaginative and 'make-believe' play is condemned by the Montessorians, but the majority of competent authorities in England and America believe it to be of the highest educational value in the early stages.

The use of small circles arranged in patterns or cards for teaching number is an excellent plan, but it is desirable that the child should be able to see the relations between the

numbers: e.g., if we take away 2 from the 5 pattern it should leave a pattern identical with that on the 3 card. Some teachers prefer to have numbers grouped in patterns, and they do not mind whether these relationships occur.



Some Number Groupings

As practically everything in the child's life can be pressed into the service of counting there is little need to elaborate the matter here. A doll's tea service for at least 6 'people', a nest of cubes, various counting games with coloured counters, sets of toys for playing shop, post-offices, 'bus-conductors', toy animals and farm-yards all have their uses. The teacher would do well to make a list of simple apparatus which can be made in the 'handicraft' work of the school at this and at later stages. The 'didactic apparatus' of Montessori (stairs, skittles, etc.), Tillich's bricks and the ball-frame (abacus) all have their points of interest and can be improvised or made in the handicraft centres. Most educational supply firms sell apparatus for the teaching of arithmetic in the earliest stages, and here we have a veritable *embarras de richesse*.

Of particular interest is a wall-board for showing the relation between the figure, the pattern, the word and a

string of balls of the same number. This can easily be made from a board 4 ft. by 2 ft. The figures, words and patterns are painted boldly on separate squares or rectangles of wood which are hung by rings on the appropriate hooks and so grouped that each set for any number from 0 to 9 is in a vertical column. The wooden balls may be strung on cords. The use of colour in this (as in other apparatus) adds to the attraction of the device, but the teacher must see that the various pieces are never on the wrong pegs. The pieces may be given to children and they can be asked to replace them correctly.

The next step is simple notation. 'In the past it was usual to proceed systematically to the analysis of numbers up to 10, then from 10 to 20 and so to establish the simpler arithmetical operations for small numbers. The experience of schools, however, which set children to experiment with counters, beads, sticks, or pebbles shows that this formal treatment is not necessary. The addition and subtraction tables are learnt as effectively and much more quickly.¹ Nevertheless place value always gives some difficulties. Teachers will benefit by reading such books as *The Story of Arithmetic* by Susan Cunningham in this connection; and they will see the attempts which were made, in the evolution of the subject to find a satisfactory notation. Our decimal system owes its origin to finger and toe counting, and there can be no objection to children counting, adding and subtracting on their fingers for a time, if a traditional systematic method is shown to them. However, it is necessary to take care that in the process of learning arithmetic there do not arise superfluous operations which waste time and bar the way to future progress. Practice should be given in setting down numbers in the appropriate hundreds, tens and units columns, etc., reading the numbers and writing them in words. The nature of these numbers can be shown by using

¹*Infant and Nursery Schools Report.*

bundles, each of which contain ten sticks. The horizontal abacus with each line of balls coloured differently will also prove useful and the child will be able to perform simple subtraction and addition by his own experiments. The child should see that figures possess place value as well as face value, and that the values can be altered by changing their position with regard to other figures. The first use of zero (meaning empty) was a notable milestone in the history of arithmetic. If the teacher carefully prepares the way, the child will realize that it is not sufficient to leave the tens column empty if there are no tens, or confusion would result. A line was first tried in the column but this was often mistaken for other figures; subsequently a dot was used, then a circle was put round the dot, and finally the circle alone remains. The idea of zero haunts all mathematics like that of infinity. It has great importance. The 'zero' question must be kept in mind throughout the whole range of elementary mathematics, for it cannot always share the properties of an ordinary number.

Addition of simple numbers may be performed by using Tillich's bricks, number rods, counters or the ball-frame (abacus). Experience leads us to believe that writing in arithmetic should not be commenced too soon—and, indeed, throughout all early work in arithmetic we must not expect perfection in too many operations at the same time. It has often been stressed that all early mathematical work should be concrete. The time has surely come to discuss carefully what is meant in this connection by 'concrete'. It is possible for a child to get so used to adding 2 marbles to 5 marbles, and taking 3 apples from 7 apples that he has found the words a stumbling block to progress. There can be no objection to the use of simple numbers without names of things attached to them.

The child should only be asked to do problems of his own construction, for it is clearly too much at this early stage to

expect him to read with understanding, do arithmetic and write good figures and words. Children should be allowed to construct their own tables of combinations and simple addition and subtraction tables. The commutative properties of simple addition and multiplication which we accept without question (very often because we have never thought about the matter at all) are not obvious. The child may know that $6 + 3 = 9$ but does not grasp that it follows that $3 + 6 = 9$.

The methods of subtraction will be dealt with more thoroughly when we come to consider Primary School work. In Infant Classes the idea of simple 'taking-away' may be approached by simple experiments with the usual apparatus. It is a good plan to give examples in subtraction while still working at addition, and to keep the problems in addition slightly more difficult; but at this stage the work should contain no hint of the formal subtraction methods which will be learnt later. Counters, bead-bars, the number-group cards, and later, simple cards with equations using both the idea of simple subtraction and inverse addition may be used.

Take away
$10 - 7 =$
$10 - 4 =$
$10 - 3 =$

etc.

Fill in
$9 - \quad = 4$
$7 - \quad = 2$
$4 - \quad = 1$
$6 - \quad = 4$

etc.

Fill in
$\quad - \quad = 5$
$\quad - \quad = 5$
$\quad - \quad = 5$
$\quad - \quad = 5$

etc.

It is sometimes overlooked in teaching early work that the common symbols and operators $+$ $-$ $\times$ $\div$ come into use without giving them a name and allowing practice to show their meaning. Careful explanations should be forthcoming.

When the child has mastered the addition of numbers up

to 20 he is ready to start multiplication; it is not advisable to start before, as multiplication is best approached through continued addition. In referring to the history of arithmetic we see that repeated addition persisted for many years before the much more difficult process of multiplication was discovered. In the Old Testament there is evidence of continued addition but little of multiplication.

In a process of continued addition, to produce a sum which to the child is a large number, the obvious advantage in using the shorter means of multiplication will be apparent. Care must be taken to see that the child can state and write in words an arithmetical expression containing a multiplication sign. Simple practical examples will suggest themselves to the teacher. Counters, ball-frames, square boxes containing small balls arranged in rectangular patterns, pictures of small numbers of animals (four paws, two eyes, two tails), flowers (with a given number of petals each), and other similar things can be employed.

In the Infant School the amount of multiplication performed will be quite small and there will be no attempt to develop it systematically in the form of tables, beyond multiplication by two or three. Simple division can also be taken before the child reaches the age of eight, and even at this early stage the ideas of measuring and sharing may be noticed. Here again simple bead-bars, counters and finally picture cards of apples, pennies, marbles, etc., for sharing will be useful.

The Infant School can also undertake the important work of testing, correcting and extending to some extent the child's earliest notions of money, capacity, a circle, a square, a straight line, etc. In the words of the Infant and Nursery Schools Report: 'The child's natural approach to arithmetic and mathematics is through the fitting of wooden inlays of various geometrical shapes into their respective frames, and through construction in which he learns to recognize shape

and the importance of measuring, through play in which he imitates grown-up activities such as shopping and playing at trams and through number games, such as nine-pins or throwing at a target. Through these activities he becomes interested in counting, measuring and calculating. . . . The range of work that is practicable will naturally vary with the capabilities of individual children. No uniform standard should be expected, but each child should be allowed to go as far as he is capable of going at his own rate of progress. Experience suggests that most children will, before the age of eight, have acquired at least the power to work straightforward sums in the fundamental processes and in money, provided that the numbers are not too large, and will have acquired some knowledge of shape through personal experiment.' All the work should make an appeal to as many sensory activities as possible, it should be dynamic and make use of the child's own inquiries and experiments; and long periods devoted to arithmetic should be avoided. As this account of work up to the age of eight is necessarily rather slight, a reading list referring to arithmetic at this stage is given at the end of the last chapter in this book.

CHAPTER III

PRIMARY SCHOOL ARITHMETIC

OWING to the traditional position of the '3 R's' in public elementary education in this country a considerable proportion of the whole of the school time has been allocated to arithmetic, even to the extent of five hours a week; and moreover, this was taken from the periods of the day which seem to be most favourable for concentrated work. Also, arithmetic has occupied a prominent place in the transfer and scholarship examinations taken at the age of eleven, often, indeed, to the extent of 50 per cent. of the written examination. It is possible to drill children so that at the end of the Primary School course they will have 'absorbed' a quantity of arithmetic which is in excess of their natural capacity. High standards at transfer examinations are often two-edged swords. The teacher, parents and successful child are pleased, and the Secondary and Modern Schools welcome the good 'grounding' which their new pupils have received; but forcing arithmetic to a state of mechanical perfection and divorcing it from real interest and human considerations has often rendered it distasteful to a child when subsequent work is done. Sometimes the work has not been consolidated and the standards obtained are artificial. A child can usually be taught to perform a process, without doing it very intelligently. For instance, it is not necessary for the child at first acquaintance with subtraction to understand completely the reasons for the process but he should, at least, know how to use it intelligently. During the primary school life the arithmetic course will centre round the first four rules—addition, subtraction, multiplication and division, put to a multitude of uses.

The position of mechanical work will require careful con-

sideration. From the time of Cocker's famous arithmetic (which, despite the fact that it became a by-word in the English language, helped to kill intelligent arithmetic for many decades) the subject became entirely mechanical. There followed the inevitable reaction against this, and the mechanical gave place to the artificial. 127×11 is hardly made more practical or interesting by turning it into 'find the price of 127 books at 11d. each'. Indeed, the simplicity of the statement in figures is obscured. Mechanical work, which is of considerable importance, has been given a definite and very useful place in arithmetic teaching. In reasonable quantities mechanical work is enjoyed by the children, for it satisfies their 'rhythmic' instinct, there is delight in getting the right answer quickly and there is joy in the increase of speed which results from the ever-growing mastery of the process. Perfection in mechanical work leaves the mind free to grapple with larger problems. A pianist cannot give his thoughts to the interpretation of a piece of music if he is troubled with difficulties in the fingering of a scale passage. Sometimes we expect 'to kill too many birds with one stone' in teaching. Formerly, at an early stage in English composition we required of our children good writing, grammar, punctuation, imagination, form, style and spelling; and in a similar way in early arithmetic we expected neat presentation, good figures, reading with understanding, logical thought, the remembering of various units and their relations, and accurate mechanical work. Short periods of ten minutes or a quarter of an hour devoted to mechanical work are often the means of introducing an acceptable variety. Work of a mechanical, repetitive nature, though dull to the teacher, is not always dull to the child.

Multiplication Tables

Most authorities on the teaching of arithmetic believe that the multiplication tables should not be taught in the

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120

order of the natural numbers. So that each table shall be better connected with the preceding one it has been suggested that the order should be 2,— 3,— 10,— 5,— 4,— 6,— ‘times tables’, or 2,— 3,— 4,— 6,— 10,— 5,— times tables. The tables may be built up by using counters and bead-bars. It is very interesting for the child to arrange the

natural numbers up to 120 in the form of a rectangle, and by straight lines drawn with coloured crayons to mark out tables of 'factors' which divide into various numbers.

The table shows 9.s, 11.s, 5.s, and 10.s marked out, and the same idea may be worked for 2.s, 3.s, 4.s, 6.s, 7.s and other less simple numbers. Later this will lead to an idea of prime numbers and 'the Sieve of Eratosthenes'.

Multiplication tables should be learnt so thoroughly that the results should come automatically—almost as reflex actions. If a child has to pause to think carefully that six nines are fifty-four, he does not know his tables sufficiently well. It is necessary to conserve the higher mental processes for the solution of problems. Badly learnt multiplication and addition tables will destroy accurate and quick arithmetic throughout the school course. It is therefore necessary to say a little about the best modes of learning the tables. Formerly, tables were learnt by daily drill in a concerted and noisy chanting by the class for 10 minutes or a quarter of an hour. This has several objections. Throughout our teaching we must see that superfluous associations and habits are not set up. The 'chanting' method is productive of the child who cannot say that $7 \times 9 = 63$ without going through the early part of the 7 times tables. This method also assumes that all children are capable of working at the same speed; the speed of 'chanting' is not the optimum speed for effective fixation, and as the conative aspect of the process tends to be lost, the mind may be wandering while the lips are repeating the tables. Unless there is real attention to what is being said the process has no value. Some teachers have recommended individual muttering of the tables, but apart from the unpleasant sound which this creates it is not ideal. Repetition there must be, but it must be leavened with variety, and any process is useless without mind concentration. The 'clock method' is also a stimulating change. A large clock-face with Arabian numerals and one hand, is

placed on the blackboard, and a figure (say 7) is put in the middle of the face. The hand of the clock is moved to various figures and individuals are asked to give the answer ($7 \times 8 =$, $7 \times 2 =$, $7 \times 12 =$, etc.). Tables may also be learnt by the quick working of a large number of sums depending on a particular table.

Children will be interested to construct a multiplication square containing a summary of all multiplications of integers up to 12×12 . Also, by drawing straight lines, etc., in the number rectangle they will be able to pick out sets of the multiples of various numbers. Practice in the inverse process (7 into 21 is 3; 6 into 30 is 5, etc.) is also useful.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

A multiplication Square epitomizing the multiplication tables.

Addition

Points in early addition may be summarized as follows:—

(1) Although there can be little objection to counting and performing very simple addition by the use of the fingers, this should be discontinued as soon as possible and no other superfluous habits should be allowed to creep in. In this respect there are more pitfalls in addition than in multiplication.

(2) The simplest form of addition seems to arise from adding by units, and this often gives place to a method whereby the addends are split up so as to recombine to form tens. Sometimes there is hunting in the column for numbers which when added together form tens, but this imposes an unnecessary mental strain unless the paired numbers come together. If the 'addition table' has been thoroughly learnt there should be no difficulty in adding the figures seriatim. The words mentally 'spoken' should be cut down to the minimum or otherwise progress is not only retarded but there is time for the memory to fail in passing from one number to the next, which is a common difficulty in addition; e.g., in adding 37, 19, 24, 48, 13 the child should say mentally 7, 16, 20, 28, 31. The carried figure should be written down (if at all) at the bottom of the column to which it belongs. It is recommended that having arrived at the 31 the 1 should be written down and the carried 3 stressed mentally.

(3) The working of all sums should be checked by adding from the bottom upwards as well as from the top downwards. Horizontal as well as vertical addition, particularly in simple cases, should be undertaken.

(4) The addition of long columns raises several problems. Attention soon lapses, and memory of the numbers already added fails. It is necessary to encourage the child to pause and relax at certain intervals, and to rest at the end of each column. If in endeavouring to add a column of figures a child has become confused at a certain point, he should be

encouraged to pause at that point while 'holding' it with his pencil and to make a conscious effort to remember the partial addition.

Subtraction

The three processes of subtraction in use are called: (1) Decomposition. (2) Method of Equal Addends, and (3) Complementary Addition.

(1) Decomposition.

This method is still very popular, presumably because it is the easiest of the methods to understand, but it is far inferior to the method of equal addition where speed and accuracy are concerned. An example will demonstrate its use:

Take 468 from 635.

$$635 = 6 \text{ hundreds} + 3 \text{ tens} + 5 \text{ units.}$$

$$468 = 4 \text{ hundreds} + 6 \text{ tens} + 8 \text{ units.}$$

as 8 cannot be taken from 5 it is necessary to decompose one of the tens.

$$635 = 6 \text{ hundreds} + 2 \text{ tens} + 15 \text{ units.}$$

and as 6 cannot be taken from 2 it is necessary to decompose one of the hundreds to tens. Finally:

$$635 = 5 \text{ hundreds} + 12 \text{ tens} + 15 \text{ units.}$$

$$\text{Subtract } 468 = 4 \text{ hundreds} + 6 \text{ tens} + 8 \text{ units.}$$

$$1 \text{ hundred} + 6 \text{ tens} + 7 \text{ units.}$$

This method may be illustrated in practice with a flat tray divided into compartments marked H. T. U. and containing respectively cylindrical bundles containing 10 bead-bars each with ten beads, bead-bars with 10 beads each and separate beads. The necessity of decomposing the bead-bars into 10 beads, and the 10 bundles into bead-bars is readily appreciated by the child, when a suitable number is being subtracted. In this, as in other elementary processes in arithmetic, the child should learn to economize in the num-

ber of words used mentally during the process; and in this particular instance the use of the words 'borrowing' and 'paying back' is both misleading and superfluous.

(2) Equal Addends.

This method is based on the fact that if both quantities are increased by exactly the same amount their difference is unchanged. A very simple example will illustrate how the method works.

$$\begin{aligned} 10 - 6 &= 4 \\ (10 + 4) - (6 + 4) &= 4 \end{aligned}$$

The 'age' illustration is also acceptable to children. Kathleen is 8 and John is 3. In 10 or any other number of years' time Kathleen will still be 5 years older than John. It is not necessary to explain the working of the method fully, provided that when children come later to use the process they will return to an examination of its action. Indeed, most authorities on the teaching of arithmetic would prefer a good method applied intelligently but without a full understanding of its inner workings, to an inferior method the theory of which may be thoroughly grasped. In practice the method works as follows:

From 456

Take 279

9 from 6 is not possible. Add 10 to the units in the top line (minuend) and 1 (ten) to the tens in the second line (subtrahend).

9 from 16 leaves 7, put down 7 in units column, 8 from 5 is not possible. Add 10 (tens) to the 10 of the minuend and 1 (hundred) to the hundreds in the subtrahend. 8 from 15 leaves 7. Put down 7 in tens column. 3 from 4 leaves 1, put down 1 in hundreds column, i.e., the problem becomes

4 hundreds + 15 tens + 16 units.

3 hundreds + 8 tens + 9 units.

1 hundred + 7 tens + 7 units,

The method may be well illustrated as before with the tray divided into three compartments and beads, bead-bars and bundles (tens) of bead-bars.

(3) Complementary Addition.

This is really another way of regarding the process of subtraction. If $a - b = x$ then $a = b + x$, or if in a simple subtraction the subtrahend is added to the difference (the answer) the minuend is given. This method relies on a good automatic knowledge of the addition table or table of combinations. A single example will suffice to show the method in operation.

$$\begin{array}{r} 423 \\ 176 \end{array}$$

$$247$$

What must I add to 6 to give 13? Put 7 in the unit column (now $13 = 1 \text{ ten} + 3 \text{ units}$ therefore add 1 (ten) to 7 in subtrahend). What must I add to 8 to give 12? Put 4 in tens column. Add 1 (hundred) to 1 in subtrahend. What must I add to 2 to give 4? Put 2 in hundreds column.

Experience has undoubtedly shown that of the three methods that of decomposition is ultimately the least satisfactory. Actually, despite the rather long explanations here given, methods (2) and (3) can be quickly employed according to the demands of the problem. Both in the cases of subtracting money and ordinary numbers, the examples which we get from everyday life do not favour the method of decomposition for the 'round' number more frequently occurs in the top line (minuend), e.g., giving change for a pound note, subtracting from 10,000 etc.

Multiplication

Multiplication will proceed by the working of a number of examples to fix the commutative law of multiplication

(e.g., that $3 \times 4 \times 5 = 5 \times 3 \times 4$, etc.,) and also to give the idea of factors. A child should not only know, so thoroughly that it is practically automatic, that $6 \times 9 = 54$, etc., but when presented with ordinary numbers up to 100 with a little thought he should be able to give pairs of factors and later all the factors. The order of development in teaching the subject will be:

(1) Multiplicand less than 100. Multipliers less than 10, e.g., 34×4 .

(2) Multiplicand between 100 and 1000. Multipliers less than ten, e.g., 467×7 .

(3) Multiplicand between 100 and 1000. Multipliers between 10 and 100.

Multiplication should take place from *left to right*.

$$\begin{array}{r} 467 \\ 87 \\ \hline 37360 \\ 3269 \\ \hline 40629 \end{array}$$

The reasons for insisting on left to right multiplication are (a) The first line of the product gives a rough check of the final answer.

(b) When decimals are multiplied, or when contracted, methods and approximations are required, there is an obvious advantage.

(4) Cases with larger multiplicands and multipliers, but with not more than five figures in the multiplicand and three in the multiplier.

'In all written work attention should be paid to its arrangement. Schools in this country have in the past rarely attached sufficient importance to the form in which arithmetic computations are recorded and the written

work is often nothing more than a succession of rough notes of computations which form part of the mental argument and are put down on paper, better because they are too elaborate to be performed mentally or because the results of the various interim operations might otherwise be forgotten. Clear thinking is essential in arithmetic and the habit of clear thinking is assisted greatly by the use of a logical form of arrangement of written work. A form for arithmetical work which follows the mental steps of a clear thinker, and has regard to its usefulness at a later stage should be prescribed.' (*The Primary School Report*.) This cannot be urged too strongly.

When children have obtained sufficient practice with long multiplication they will discover for themselves little devices for short methods. Multiplications by 10, 100, 10,000 are obvious. Multiplication by such numbers as 5 (by adding a 'nought' and dividing by 2), 25 (by adding two 'noughts' and dividing by 4), by 11, 99, etc., will be discovered when the children have a larger acquaintance with the simple properties of number. No opportunity should be lost of encouraging children to experiment for themselves to find these properties. Actually, children take a greater interest in pure number than is generally realized. The numbers of houses in streets, of motor-cars and tickets are often the basis of some interesting experiments with numbers. A distinguished professor of mathematics once said that his interest in the subject began when as a boy he used to play in the street, and perform mental gymnastics with the door numbers. It is perhaps better not to force a knowledge of this aspect of the subject and of its applications. We shall refer to it again later.

Division

Although not quite as much stress is now put on the twofold nature of division as in the days when the inspired

German text-books first began to influence English arithmetic teaching, nevertheless, clear ideas of measuring and sharing (Quotition and Partition, 'messen und teilen', etc.) are very important.

(1) Measuring, Quotition, grouping, dividing by:

I possess 80 marbles. To how many boys can I give 5 each?

How many times can I cut pieces each 2 ft. long from a rod 12 ft. in length?

How many shillings in 264 pence?

(2) Sharing, Partition, dividing among, etc.:

I share 80 marbles amongst 16 boys. How many do I give to each?

Cut a rod of 12 ft. into 3 equal parts. What is the length of each part?

What is a twelfth of £1?

The differing nature of these processes may be demonstrated with a ruler and piece of string, and by other practical considerations. It will be seen that there are radical differences—and not only when the question of a remainder arises.

Authorities are divided as to the best order in which certain of the topics in arithmetic should be taken. Some argue that as sharing involves the idea of fractions it should therefore be postponed until some knowledge of the latter subject has been acquired. On the other hand, from a practical point of view children will probably have met with the process of sharing before that of measuring. As one of the necessities of all elementary mathematics teaching is to return frequently and quickly to past work (both for the purposes of revision and also for reconsidering it in the light of subsequent knowledge), this point need not worry us unduly. In the early stages experiments with counters,

bead-bars, marbles, and the ball-frame will serve to show the nature of the processes. Special apparatus has been used by Dr. Montessori for illustrating division, but over-elaborated apparatus may prove a stumbling-block at a slightly later stage.

Short Division and Long Division

In the past, many teachers of arithmetic have postponed the teaching of long division; and indeed, it is certainly the most difficult process in the first four rules. Nevertheless, as the subject is of such great importance, it is better to commence it as early as possible and to use simple apparatus to demonstrate its nature.

(1) Short Division. Divide 3785 by 12.

$$12 \overline{)3785}$$

315 and 5 remainder.

N.B. 3785 is called the dividend, 12 the divisor and 315 the quotient. It is not correct to write Ans. 315 + 5 remainder, for there are 315 groups of 12 and the remainder is 5 units.

(2) Long division. Notice carefully that the quotient is written immediately above the dividend with its digits in corresponding positions.

Divide $682,159$ by 834

$$834 \overline{)682,159}$$

667 2

1495

834

6619

5838

781

Answer 817, and 781 remainder.

The Italian Method for Long Division

This is definitely upper-form work and then only for children who have acquired considerable facility in handling figures and who can use the method of complementary addition for performing subtraction. It differs from the normal mode of performing long division in that the multiplications and subtractions are performed mentally and only the remainders are written down at each stage. This involves a saving in the number of figures written down.

It is doubtful whether it is worth spending much time in teaching division by factors. Not all numbers factorize, and even when the divisor can be split into simpler numbers it is debatable whether any advantage accrues, particularly as the remainders have to be totalled carefully. In modern arithmetic teaching it is better to concentrate on essentials, and in this case to spend the time in perfecting long division through a carefully-graded series of examples. Nevertheless, a single example is given here to show the process in operation.

Divide 36,188 by 105 ($= 3 \times 5 \times 7$)

3)36,188

5)12,062 (groups of 3) + 2 units remainder.

7) 2,412 (groups of 15) + 2 groups of 3 remainder.

344 (groups of 105) + 4 groups of 15 remainder.

Total remainder $2 + 2 \times 3 + 4 \times 15$

ANSWER: 344 and 68 remainder.

Summary of Division

(1) When dividing by 12 or less, or 'easy' numbers such as 100, 4000, 120, it should be insisted that children use short division.

(2) A logical way of setting down the working of each division 'sum' should be demanded from the first. This will prevent errors both in the setting down of the quotient and the remainder.

(3) Dividends and quotients which contain 'noughts' are productive of many errors. These should be carefully approached and attention paid to the use of correct columns.

(4) Checking should be done by reversing the process, i.e., multiplying the quotient by the divisor and adding the remainder which should give the dividend; or again, dividing the dividend by the quotient which should give the divisor and leave the same remainder.

(5) Tests of divisibility by 2, 4, 3, 5, 9, etc., should be acquired gradually, and employed for checking purposes.

Weights, Measures, Money and Time

Early arithmetic will have three broad aims in view: firstly, to understand the nature of the measurements which we make, the units we use (and the relations between them) in money, distance, area, capacity, weight, time, etc.; secondly, to be able to use intelligently the early mechanical processes of arithmetic and to select the most appropriate method; and thirdly, to apply these processes skilfully and accurately to useful problems. (Analogous aims may be traced in quite advanced work in mathematics and the dependent sciences.) In the past the second aim has been stressed to the exclusion of the other two.

Tables of weights and measures should be cut down to the minimum but the chief ones should be learnt thoroughly—as thoroughly, indeed, as the multiplication table. Important though this is, it is the least part of the matter. Weights and measures should be approached practically and historically:—practically, because they are used so constantly in our lives, and historically, because that is the only way we can really understand them. Children living in foreign countries are spared the worries of our traditional systems and much time is saved. Although the advantages of the 'Metric System' may be anticipated in the Primary School, a full treatment will be reserved until later. Children commence

school with various scraps of knowledge in relation to weights and measures. The tables should be built up experimentally. Our present tables are the results of efforts, through many years, to reduce the countless systems which were often entirely unrelated to one another. Tradition and custom are still too strong for us to make a 'clean sweep' of some of the remaining tables and adopt the more logical Metric System. When this is dealt with in the Senior School and applied to scientific measurements, its ease and superiority are at once evident. Indeed, it is interesting to search in the history of chemistry or physics to find measurements and calculations performed in the English system, and to observe how clumsy and difficult they are.

Ideas about money can be introduced by playing at 'Shops' or 'Banks' with real money. Children are always interested in coins of other days, such as a crown, a groat or a gold sovereign. An outline of the history of our coinage will be of interest if it is illustrated with real coins and pictures. The words pound, shilling, penny, crown, florin, farthing, etc., will appear from their historical use. Children will probably ask for the derivation of the slang names given to certain coins, and although they should not be encouraged to use these, the origins of the names are often interesting. English literature and history are full of references to the names of coins, many of which are no longer used. The question of 'legal tender' with regard to the use of farthings and silver is interesting and should be mentioned. In the Senior School the child will consider the coinage of other countries in particular, France, America, Germany and Holland; and will be interested in suggestions which have been put forward with regard to the possibility of a metric system in England. (Already the florin is $\frac{1}{10}$ th, and a farthing practically $\frac{1}{400}$ th of a pound, a new coin of value ($2\frac{1}{2}$ d. nearly) would be required for $\frac{1}{100}$ th of a pound, and five of these would make one of our present shillings.)

Not many decades ago, almost every trade and profession had its own system of weights and measures. In the case of the Worshipful Companies these systems were often a part of their jealously guarded secrets. Even to-day when we can compare all systems of weights and measures there are still secret societies—the traditional outcome of these worshipful companies—which use their original terms for measuring, and continue to make great use of the historical and mystical significance of quantities and numbers. The sailor still uses fathoms for finding depths, but on the other hand, we are at least spared the difficulty of going into a draper's shop for an ell of cloth. Although children will be pleased to find so much history reflected in our inconvenient systems of weights and measures, real interest will arise in knowing their nature and how to use them practically. Rods, poles and perches, if used at all, can be made intelligible in the school garden; acres, roods and other square measures by reference to playing fields, always keeping in mind their historical derivation. Measurements of capacity should be dealt with carefully. It is useless to say $4 \text{ gills} = 1 \text{ pint}$; $2 \text{ pints} = 1 \text{ quart}$; $4 \text{ quarts} = 1 \text{ gallon}$; $2 \text{ gallons} = 1 \text{ peck}$, etc., if the uses to which these quantities are put in everyday life are overlooked. Here some simple apparatus is necessary, but it is such that it can be found in every household. Children will delight in experimenting with sand or water to compare the capacities of various containers. Wine-bottles, medicine bottles, 2-gallon petrol-tins, 1-gallon oil-drums, an old 20-gallon petrol-tank, a pint mug, a school milk bottle, a watering can, a small barrel, a cup, various 'tin' containers, jam-jars and many other things can all be pressed into service. Although this work will be done more thoroughly in post-primary school days, children in their tenth and eleventh years will be interested in estimating (wrongly called guessing) the capacities of various containers. They should get to know the 'feel' and size of a

three-gallon watering-can, a gallon tin which always seems surprisingly small, a pint mug, etc. Owing to mixed fragments of information which they have picked up chiefly through accompanying their parents on shopping expeditions, they may want to know why milkmen often call a half-pint a gill. Also, they will have seen capacity measures in use for estimating quantities of strawberries and other soft fruit, shrimps and coarse seeds (pints), peas in their shells (pecks) and other dry measures. They will realize that this method is quicker than weighing but is open to the objections that weight cannot be constant if the articles bought show a disparity of size at different times. Children are fascinated when given an opportunity to weigh things, either with spring-balances or with the balances and weights used in the kitchen. Weights beyond a stone are often too heavy for children to lift, but they should be shown other weights. The local Weights and Measures authorities may often be in a position to lend obsolete weights. Children are interested in finding the weights of themselves and of common objects such as chairs with a spring balance or other weighing machine. Children should know that 1 pint of water weighs $1\frac{1}{4}$ lbs. and 1 cubic foot of water 1000 ounces. Here again is work which must be extended to the Senior School. The words which children use to describe the simple physical properties of things need reconsidering, correcting where necessary, and extending. Only careful individual questioning will elicit from them the notions which they attach to the terms they use. For instance, the word 'heavy' may be used to mean 'weighty' or 'dense', i.e., actual or comparative 'heaviness'. Exact ideas of density are for the Senior School, but they are by no means as difficult or as tedious as elementary text-books of physics have tried to make out in the past. A boy given a cubic inch of cork in one hand and a cubic inch of lead in the other will readily say that the lead is the heavier. When given a piece of cork

large enough to make a life-belt, he will say that it is heavier than the cubic inch of lead. By careful questioning he will see that lead is much heavier than a piece of cork of the same volume. Children should be encouraged both at home and at school to get used to weights, and by practice to be able to estimate them roughly. In fact, the Boy Scout requirements of judging heights, distances, and other measurements might well be used in Primary and Senior Schools. Children can estimate a mile in country walks or in their journey to school, and shorter distances from the length of a street, the school playground, the swimming bath, etc. As there are various ways of travelling a mile: 'dawdling', walking, running, riding a bicycle, getting a ride in a car, travelling in a train, and in fast aeroplanes or racing cars (about which they will have read) the general idea of speed will come quite naturally.

The important question of time is often overlooked. A large clock-face with movable hands (or an old clock) will be necessary. A fuller treatment of time will come in Senior School work, with particular reference to time in astronomy, engineering and science. In the Primary School the general ideas of telling the time, the meaning of the terms 'fast', 'slow', 'losing', 'gaining', a.m. and p.m., the 24-hour day and simple 'reductions' will suffice. Much material for use in simple problems on time can be obtained from *Bradshaw*, *A.B.C.* time-tables, 'bus time-tables, B.B.C. programmes, etc. A day for which a time-table is made tends to be an efficient day—whether it is used for business or leisure. Many adults are inefficient in their work and spoil the effectiveness and enjoyment of their leisure hours by not troubling to divide up their time in a satisfactory manner. Work in the Primary School may also include ideas of a decade, year, season, month, week, day, hour and second. A simple account of their historical and astronomical significance will be given later. The old rhyme: 'Thirty days hath

September' is worth learning. Work with calendars with rotating numbers, or the use of 'everlasting' rubber date-stamps for printing dates is a source of fascination to children. (See also the section on Time.)

A considerable part of the arithmetic of the Primary School is commonly called 'reduction'. In the past it has been greatly exaggerated in the extent of the processes performed, impracticable problems have been used and the work has not been fortified by practical experiment with counters, real or cardboard money, hollow cubes to be filled with sand and other simple apparatus. Some of the artificial problems of reducing millions of inches to miles are absolutely useless.

'The practical utility of reduction is to simplify computation and it should be confined to examples of this nature. It is less trouble to deal with 3 yards 1 foot 6 inches as $3\frac{1}{2}$ yards or as $10\frac{1}{2}$ feet than as it stands, or than as 126 inches. Reduction need never include more than three consecutive units—pounds, shillings and pence; yards, feet, inches; gallons, quarts, pints—and should take two forms (*a*) reducing from larger units to smaller (reduce £2 11s. 6d. to sixpences) and (*b*) expressing smaller units as a fraction of a larger unit (Express 11s. 6d. as a fraction of £1).'—*The Primary School Report*.

Oral and Mental Arithmetic

Oral arithmetic is important for many reasons:

(*a*) It is a way of varying the routine of the arithmetic lesson. Quick animated questions directed to all corners of the class will bring immediate attention; or again, the answers may be written down on paper.

(*b*) It demands powers of careful listening, visualization and quick thinking. This is useful when longer problems are worked in the ordinary way, for some of the slighter parts of the problem can be solved quickly by mental work. Investigations in a number of schools revealed that poor

results in oral arithmetic were not entirely due to faulty imagery and lack of practice in quick thinking, but were often the result of bad listening. An intensive course of ear training resulted in improved oral arithmetic! Although hurry in the early stages of teaching arithmetic is absolutely fatal, it must be stressed that speed in performing the simple mechanical processes is an ultimate necessity.

(c) Oral work is a means of introducing a new subject through the medium of very simple examples.

(d) It is also a means of quick revision, and of giving the teacher a rough estimate of individual progress, but it must be remembered that some otherwise excellent children show no aptitude for oral arithmetic.

Some Final Remarks

Many teachers will broach the subjects of fractions and decimals in Primary Schools. Work should be very easy; and the matter has been mentioned in a later chapter.

In deciding on methods of teaching and framing syllabuses the teacher cannot be guided by fixed rules. In many schools six-monthly, terminal or even six-weekly schemes of work can be mapped out and the child promoted according to ability. Often in city and town areas it is possible to organize three or more streams of work for A, B, C, and even D children. In country schools this is often impossible owing to the smaller number of children and the few teachers available in the school. The difficulties here are extreme, for there is a temptation to neglect the C children entirely and to concentrate on the A children, especially if these are going to sit for scholarship or transfer examinations; or again, these children are allowed to make their own way and much time is spent with the weaker children. These difficulties are accentuated when it is found that the backward or mentally defective children have had no special provision made for them. The teacher has to divide the class

into several sets, and rely on a good text-book of the 'self-help' kind. Although the child's exercise-book needs thorough marking by the teacher at frequent intervals, it is possible to delegate a considerable amount of the routine marking to the children themselves. The teacher should make every effort to walk round the class systematically, while the children are working their examples. It has been said that good teaching tends to minimize the difference in attainments between the best and weakest members of the class. No statement can be less true if the class is taking arithmetic in a country school! An excellent course in arithmetic for Primary Schools is provided in the four books of *The New Freedom Arithmetic*, by Turner and Foot, published by Basil Blackwell.

The desiderata which we have mentioned above have been fulfilled. The general scheme is as follows:

	1st Year	2nd Year	3rd Year	4th Year
RULES	$+ - \times \div$	$+ - \times \div$	$+ - \times \div$	$+ - \times \div$
NUMBERS	To 100	To 1,000	To 10,000	Any size in reason
MONEY	Shillings, pence	Shillings, pence (to £1)	Pounds, shillings, pence, farthings	Pounds, shillings, pence, farthings
LENGTH	Feet, inches	Yards, feet, inches	Chains, yards, feet, inches	Miles, furlongs, chains, yards, feet, inches
WEIGHT	Pounds, ounces	Stones, pounds, ounces	Hundredweights, quarters, stones, pounds, ounces	Tons, hundredweights, quarters, stones, pounds, ounces
TIME	Hours, minutes	Days, hours, minutes	Weeks, days, hours, minutes	Weeks, days, hours, minutes
FRACTIONS			Very simple	Simple
DECIMALS			Tenths Hundredths (+)	Tenths Hundredths
CAPACITY				Quarters, bushels, pecks, gallons, quarts, pints, gills
AREA				Simple rectangles

In the *New Freedom Arithmetic*:

(1) The work concentrates only on fundamental processes and all unnecessary matter and involved examples have been discarded.

(2) There is a uniformity of plan in all books which makes for easy reference.

(3) There is a simplicity of plan. Each book is divided into two sections and these into four parts corresponding to Addition, Subtraction, Multiplication and Division, to which is added a short revision section. Adherence to the same sequence of rules and measures in both sections makes for an ease and familiarity which should result in confidence and accuracy of working.

(4) The books permit elasticity in use. The year's work is laid out so simply that teachers can quickly and easily adopt the books to suit their own needs. The schemes of work are not enslaved by the books. The order in which the various measures are taught can be altered without trouble. The division of lessons in the second section of each book into First Reading and Second Reading makes easy arrangement of work for A and B classes and the working of six-monthly promotion schemes.

(5) Simple explanations of processes with typical examples worked out at the head of each lesson. These are ready to hand for the child's own reference. Apart from the mechanical work the examples are interesting and relate to daily life.

(6) The books are printed in large type with bold figures. The necessity of this is often overlooked.¹

¹See *The Primary School*, p. 259.

CHAPTER IV

MENSURATION AND THE BEGINNING OF GEOMETRY IN PRIMARY SCHOOLS

EARLY notions of shape and size which the child has picked up from his environment will be the starting point of simple work in geometry, but these random experiences have to be corrected and put into order before they can be used. Normally there are enough interests in the life of a child of eight to eleven to stimulate him to begin a quest for some accurate ideas about shape and size. 'From seven on to puberty the thorough-going recapitulationists distinguish four successive phases which roughly correspond with the four stages which they term, somewhat fancifully, the hunting, the pastoral, the agricultural and the commercial phases respectively.'¹ Although the theory of recapitulation may lead to dangerous assumptions the suggestions made above are broadly true. All these phases are accompanied by a strong desire to construct simple things, whether with scissors, paper and paste, with clay, sand, mud or with constructional toys such as Meccano. The child's history lessons are made more interesting by making cardboard models of churches, castles and villages; his geography, literature and science are made to live in a similar manner. The girl also takes an interest in knitting, sewing, making and using patterns, and thus children soon begin to realize the necessity of making plans and taking measurements. When children see that mensuration and geometry lead to something which is useful and interesting to them, half the battle is won. The history of early geometry, or land-measuring and the simple mensuration mentioned in the

¹Dr. Cyril Burt in *The Primary School*.

Old Testament will not only prove an incentive to work, but may suggest means of approach, e.g., the old method of determining an area by finding the quantity of seed which would just cover it, is worth thinking about.

Rectangles may be built up by using inch squares, and the results applied at once to tiles, paving stones, bricks, the number of trees in an orchard each needing a square of a certain area, painting squares on a pattern, the number of square cartons (Oxo cubes, starch, etc.) which will form one layer of a 'rectangular' box, the number of square divisions in the container in which the school milk bottles are brought, other sectional packing cases for eggs, etc.

The simple ideas about the areas of rectangles and the volumes of cuboids are usually taught very thoroughly in Primary Schools, but children are also interested in a variety of shapes such as that of the crescent moon, and teachers often fear that if they are not able to deal thoroughly with these—which is an impossible task—all work on them should be omitted. It cannot be too strongly stressed that the groundwork of arithmetic should be taught with the utmost thoroughness, but this need not prevent the pursuit of many byways for the sake of mere interest. Persistent lack of interest in arithmetic is an evil which must be prevented by every means at the teacher's disposal.

Past difficulties in understanding the nature of area have been due largely to a failure to allow the child sufficient opportunity of handling things. He can be drilled to think of area as the result of a multiplication sum and yet have no grasp of the nature of surface measurement. Filling in outlined areas with a coloured wash, painting a model or 'creosoting' a fence will help to form what is sometimes a difficult new conception. In the Infants' School, children should have had practice in handling 'shapes' and fitting them as insets in a board cut with appropriate holes to receive them. Often children find the greatest difficulty in

grasping the nature of an area from an outline. It should always be filled in with colour or ruled off in squares at the early stages. Children often find counting spaces correctly a troublesome business, as they seem preoccupied with the dividing lines. (Counting notches in 'tally-sticks' would seem to be a much more ancient and firmly established operation than the counting of spaces, which may explain the difficulties which arise when a protractor or ruler is used at an early stage.) The units of square measure should be introduced practically and applied immediately to real problems. The work should extend itself concentrically from square feet and square inches as the child's experience widens. Many examples of square feet, inches and yards may be found from the areas of carpets, the area of paper and wood for modelling and other constructional work, in book-binding and the school garden. The stock sets of problems about papering the walls of a room and drawing plans of the school premises are rather antiquated, but a few examples may be taken.

In the first case, many more examples from ordinary experience are more interesting, e.g., carpeting, staining, making a shed, a fence, a 'Cinema' screen, a hoarding, a cricket Scoring-board, laying a floor or pavement.

(It is necessary to explain what is meant by buying 'a number of *feet* of timber' or 'a *yard* of calico'.)

These everyday usages produce many misconceptions with regard to area.

In the second case, interest will be stimulated if some field or other area at a distance from the school is chosen for measurement.

A number of children find difficulties with area because they do not grasp the idea of 'squaring'. Others are not able to understand the meaning of a diagram. Children should be encouraged to hunt for their own problems on area, and

to perform simple experiments to enable them to solve some of their problems. Handwork and art yield many useful examples, and in the Primary School it is not too much to expect that the more intelligent children will begin to find distances on maps by measuring the length of the line drawn between places, or the length of a piece of thread which follows the journey as marked on the map. Such children may continue with some of the topics which are normally undertaken in the Senior School. All teachers should read various books on Pathfinding and Surveying, which deal with the requirements for the various Scout and Guide badges. The value of 'Boy-Scout geometry' cannot be overrated, and it is amazing to find the extent of the knowledge of practical mensuration which is possessed by many intelligent scouts at the age of twelve or thirteen years.

Cubic measure will take an analogous treatment. It should be investigated practically by building up a cuboid with cubic inch blocks and by making boxes which are open at the top, by folding and gumming cardboard. 'Cubic-inches' (open at the top) may be used for experimenting with sand. Tinplate models which are readily made in the metal shop will enable experiments to be performed with water. The child should discover the number of cubic inches in a pint, the weight of a cubic foot of water (1000 oz. $62\frac{1}{2}$ lbs.), the weight of a pint of water ($1\frac{1}{4}$ lbs.), the 'feel' of a cubic inch of lead, of wood, etc., the number of cubic inches in a teacup, the number of cubic feet and gallons in a tank, the number of cubic feet in a cartload, the number of cubic feet of soil to make a rockery, and approximately how many 'spadesful' and barrow-loads this represents, the number of cubic feet of air in our schoolroom, and how much for each of us, etc. Hundreds of other examples will suggest themselves to the child, who will see that he has learnt to use a new tool of great utility. Children should learn to talk about geometrical shapes and properties in accurate terms. For instance,

'perpendicular' is often confused with 'vertical' because it has always been drawn as a vertical; 'edge' is called 'side'; the words angle, corner, and point are used indiscriminately; a 'square' box may mean any box with rectangular faces and so on. Here the mathematics teacher should ask for the co-operation of his 'crafts' colleague and where the traditions of geometry and of carpentry differ, as in the use of the word 'edge', special explanations should be offered. It is not pedantic to insist that terms which are convenient and logical should be used in their conventional sense. Mathematics aims at description in an exact manner, and it relies on the use of words as well as symbols. Frequently the result of due attention to the latter is ruined by the ambiguity of the former.

CHAPTER V

MATHEMATICS IN SENIOR, CENTRAL OR MODERN SCHOOLS

IN post-primary schools, the aim is to provide for every child during a minimum period of three years after the age of eleven, some form of education appropriate to his natural abilities and potentialities. The report of the Hadow Committee in 1926, *The Education of the Adolescent*, laid down some general ideas for work in a number of school subjects for children who had finished their Primary School course. The work of reorganizing education is still going forward and has shown itself in a most satisfactory manner in the creation of a number of new well-equipped Central or Modern Schools. Even when this task is complete the future still holds many difficulties to be overcome and anomalies to be removed.

With regard to work in Secondary Schools we do not propose to say much, but it is evident that the cause of many of the school subjects could be better served by applying them to daily life rather than by regarding them as the elementary steps to later university work. The majority of Secondary School children do not go to universities, but many of them spend as much as seven years at these schools; and this ought to allow time for a broad, vital and useful treatment of almost any school subject. Even the university honours degree is not beyond criticism when it is considered in the light of a preparation for things other than research and an academic career. Unleavened by any practical, historical or philosophical considerations, the university degree is not in itself a sufficient qualification for vital mathematics teaching in Central and similar schools. It is improbable that any Central School boys, or more than a few Secondary

School pupils will become academic mathematicians. On the other hand, they all ought to become intelligent citizens, and many will take up trades, hobbies or general interests which are made more fruitful by the application of simple mathematical principles; and even in the case of the higher 'pure' mathematics, 'the history of the subject has clearly shown that the mathematical inventions which have proved to be most accessible to the masses are also those which exercised the greatest influence on the development of pure mathematics.'¹

The problems of teaching in Central and similar schools (though beset with difficulties such as the short time available for comprehensive courses, and often the lack of suitable equipment) offer many attractions. These schools have no ancient traditions to maintain, and they are quickly building up new ones; their syllabuses are not dominated by examination requirements and the classes are often taught by the younger generation of enthusiastic teachers.

It is to the credit of the Central and Local Education Authorities that to teachers in modern schools they have allowed great freedom in methods of teaching the contents of the syllabus and the time devoted to each of the various subjects. Although there is necessarily a considerable substratum of basic essentials in any syllabus, nevertheless there are many developments of the subject which will suit the individualities of teacher and taught, and also will bear reference to local conditions and industries.

The scope and nature of a school course cannot be determined by a consideration of the terminology of the school; often the term 'Senior' School is intended to refer to a school for children who are not considered capable of profiting from the work of a Selective Central School. There is no uniformity in these matters. Many districts have non-Selective Central Schools which take children of eleven of

¹Dantzig, *Number, the language of Science*.

many degrees of intelligence, ranging from some who have been prevented by circumstances from going to a Secondary School to those who would be better placed in special schools. Large city areas can often provide post-primary schools for many degrees of intelligence, and schools with biases to many professions for children of thirteen or fourteen. This simplifies to a certain extent the schemes of work in any particular school. On the other hand, post-primary schools in urban or rural areas have to do the best they can under circumstances which are often not ideal. A rough division of children into A, B and C streams is not ideally sufficient, but owing to difficulties connected with organization (staffing, time-tables, class-rooms, etc.) it is often the best that can be done. If mathematics is taught from the practical, human, civic and historical standpoints the correlation between mathematical ability and general intelligence is remarkably high. A child who does badly in mathematics but possesses a normal Intelligence Quotient, usually reflects bad teaching at the primary stage. It is perhaps platitudinous to say that in the case of a three-stream course in mathematics, it is not sufficient to dilute the A course to obtain that for C children. Not only should the three courses be different in content, but they should be distinct both in treatment and methods of approach. If it is quite impossible to separate the three streams entirely, it is better to combine the A and B courses, simplifying the former to obtain the latter, than to try to combine the B and C work.

In Central Schools there should be no artificial separation of the mathematical subjects; the powerful generalizations of algebra should grow naturally from previous work in arithmetic; geometry should evolve from mensuration, and lead to graph-drawing and trigonometry, which in turn are related to the other subjects.

The simple mathematical operations are the tools which

we can use for solving our problems. Having mastered the intelligent use of our tools our main task is to apply them to make really useful things. In the past, the tools were kept in separate boxes and their use was confined to the manufacture of artificial and not very serviceable things, but nowadays we keep them together in the same chest and use them to make real chairs, boxes and tables. Indeed, we are sometimes given a fresh problem and find it necessary to introduce a new tool which is often the logical development of an old one. We could, with some difficulty, make a round table-leg with a spoke-shave, but we have much more scope if a lathe is available! In the same way, in elementary mathematics a considerable amount of surveying is possible by drawing simple triangles, but the work is made more effective and easier by using the elements of trigonometry.

In recent years the reaction against the old mechanical arithmetic, which was often quite unpractical, has led to a reduction of the number of school hours given to the topic. Often the whole of the time between the 'Scripture' lesson and the morning recess ('break') was utilized daily for arithmetic. The arithmetic itself was monotonous enough, but it was rendered more distasteful by the invariability of the daily hour at which it was taken—a time, moreover, which represented, from the teaching point of view, the best hours of the day. Some schools have cut down their total time for mathematics to two hours a week. In parts of America this reactionary movement against the old-fashioned mathematics in favour of economics, heuristic science, logic and practical studies has been so strong that mathematics professors, fearing that their subject should lose its prestige, are doing everything in their power to popularize it. It is unfortunate that some schools in the process of paring away the inessentials of the old Victorian arithmetics have not thought fit to replace them with matters of real use and interest,

Four hours a week in boys' schools, and perhaps slightly less in girls' schools, is not an excessive allowance for mathematical studies, especially if these include work which may help the science, geography, art and crafts and other teachers. Indeed, more time than this cannot be considered exorbitant. To-day there is no need to adhere to the old plan of always taking mathematics during the early periods of each morning. With the emancipation of the subject, mathematics can find its way into the afternoon periods. A lesson based on the design of a building, a practical survey of a field, a lesson on navigation, on the history of mathematics or on the 'arithmetic of citizenship' will, if enthusiastically conducted, hold the attention even on a Friday afternoon. Many mathematics lessons fail in all types of schools by a lack of variety of treatment. There is no need for this—for even a slight amount of arithmetic technique finds more useful applications than can be exhausted in a school career. A mathematics lesson can be varied by a few minutes of revision, a little stimulating oral arithmetic, mechanical drill in 'small doses', practical work or the calculations resulting from practical work, a suitably illustrated exposition of 'civic arithmetic', or the history of the particular topics in mathematics which are then being discussed.

Mathematics teachers in Senior Schools should endeavour to obtain the co-operation of colleagues who teach other subjects. Science constantly demands much simple general mathematics in all its branches; geography on the economic side needs graphs and simple statistics; and on the side of topography and cartography, considerable applied geometry and trigonometry are necessary for surveying and map-making. History requires an idea of the time-line and often uses graphical methods of expression. The art master finds that drawing, design and architecture provide many examples for geometry, symmetry, proportion and even

mechanics. The crafts teacher likewise finds 'costing', mensuration and simple geometry are necessary for his work. Even such subjects as music (which in the days of the Greeks was regarded as a practical branch of mathematics) divides time mathematically, uses arithmetical symbols for describing its chords, arithmetical proportions in its 'architecture'; and when considered from the acoustical point of view it is replete with applications of mathematical physics. Masters teaching other subjects welcome the help of the mathematics master for they can then devote a larger part of their time to descriptive and qualitative work; and on the other hand, the mathematics teacher is able to make his subject live by useful applications, and the work of the school as a whole benefits by being correlated in a natural manner.

In the case of normal children the syllabus given in Chapter III will have been covered by the time the child reaches the Central School. Revision and extension will be necessary but this need not seriously interfere with the new work. Some further work in pure arithmetic will be necessary:¹

'(1) Multiplication and division of money, lengths, times, weights and capacities by numbers greater than 12.

'(2) Extension of the work in fractions but still without undue complexity.

'(3) Addition, subtraction, multiplication and division of decimals.

'(4) Factors, index notation, square roots (by factorization), prime numbers, the general process of finding square roots.

'(5) Logarithms.

'(6) Important concepts such as average, rate, ratio and proportion.'

¹B. of E. Pamphlet No. 101.

Much time is often wasted when a child changes schools because different methods in mechanical arithmetical processes are in use at the different stages. Uniformity of methods in the fundamental processes in elementary arithmetic implies no sacrifice of individuality on the part of a teacher. There is ample room for originality in problems to be solved and in the various ways of dividing the lesson and treating the subject matter. Teachers of elementary mathematics at all stages in every area should meet periodically to discuss fundamental processes. This will ultimately save the teacher considerable time and the child much worry.

The Central School teacher has a very wide field from which to choose, and although he will avoid any definite vocational bias in his treatment, local activities, especially where they form part of our great agricultural or maritime tradition, will suggest various modes of treatment and numerous suitable examples. It is both impossible and undesirable to make any 'hard-and-fast' rules about the construction of a syllabus. Later chapters in this book have been devoted to many of the topics in mathematics which could be taught effectively in Central Schools. In classes of normal intelligence arithmetic will lead to algebra; mensuration to geometry, simple trigonometry and graphs; and according to circumstances these subjects may be applied to various undertakings.

Very often the child may be encouraged to find his own problems, and by simple projects will gather useful mathematical knowledge, gain confidence in his powers of applying it and a strong sense of its values.

It is not to be imagined that all the Senior School topics mentioned in this book can be taught in a period of three years, but the teacher may find a selection of them useful as a guide when he frames his syllabus. The special problems of arithmetic for 'C' and defective children, and of mathematics for girls will be mentioned later. In the case of the

normal Central or Modern School course in mathematics, we may summarize our aims as follows:

(1) To provide a course which is practical and definitely useful in everyday life, and for an appreciation of the duties of intelligent citizenship.

(2) To develop that side of mathematics which will prove a useful factor in the study of elementary sciences including biology, in geography, economics, craftwork, housecraft, art and related subjects.

(3) To connect the various topics in mathematics so that the impression of a unified subject is obtained.

(4) To reconsider entirely the position of certain of some of the more traditional aspects of arithmetic, algebra, and deductive geometry with a view to the removal of those parts which are not very useful in a general practical scheme.

(5) To humanize the subject through its history and its applications to human activity—two aspects which often involve one another.

(6) Through simple books on astronomy, engines, surveying, mathematical machines and instruments and even on puzzles and the properties of numbers, to encourage a permanent interest in the subject in the later years of school life and after.

(7) To show that mathematical control makes for efficiency. To encourage the application of rough estimates and common-sense 'checks'.

The Central School course is more than an extension of Primary School work, and it need never be a simpler edition of the present-day Secondary School or university course. The main difficulty, and one which often disappoints the enthusiastic teacher, is the brevity of the Central School period, which is usually only three years or less.

In many cases during the third year of the senior school

life, children are leaving at the end of each term, and for this and other reasons it is best to draw up five half-yearly courses, together with a sufficient number of less essential and 'applied' topics with suitable individual and group work, for the last part of the third year; or in the fortunate circumstances which are to be found in some schools, for the additional fourth year.

The basic knowledge of arithmetic with which a child 'will enter a Post-Primary School at the age of eleven may reasonably be expected to include:

'(1) Addition and subtraction of whole numbers, money, lengths, times, weights and capacities without undue complexity as regards numbers and units;

'(2) Short and long multiplication and division of numbers;

'(3) The process of reduction applied to simple examples only ;

'(4) Short multiplication and division of money, lengths, times, weights and capacities;

'(5) Addition, subtraction, multiplication and division of fractions with small denominators.'¹

As further arithmetic is largely a process of extension of old work, it will be necessary to return to graded examples on the above topics from time to time for revision purposes.

Frequent sets of mixed examples such as those given for revision in the *New Freedom Arithmetic*, used as tests will determine the child's progress and reveal his peculiar weak and strong points.

The teacher will keep an eye on the practical applications of all the standard processes, and where possible introduce some of the topics mentioned in the sections on geometry, mensuration and drawing, trigonometry, surveying and navigation, general physics, the history of mathematics

(though no course can include more than a selection), and his scheme will depend on

- (a) The total time available.
- (b) Environmental considerations.
- (c) The intelligence of the child.
- (d) The natural interests of the teacher.

It is a good plan for each teacher in a modern school to have for reference a copy of the schemes of work in all subjects throughout the school. Useful correlations may be found in handicraft, art, geography, citizenship, science and even in history and music.

It has always been assumed that girls have less aptitude for mathematics than boys, and although recent investigations appear to show that this is true to a small extent, it is necessary to remember that usually girls have not had the same time or opportunity for the subject. The average time devoted to mathematics in senior girls' schools to-day is three hours or less. The demands made by housecraft, domestic science and related subjects on the 'tools of mathematics' are not very considerable; if the utilitarian aim alone determines the course in mathematics, and if the girl's after-school life is to be limited to the home, the simple groundwork of arithmetic and mensuration will suffice. Surely this is not a satisfactory way of regarding the subject! Naturally, it is probable that in later years a girl will need the applications of mathematics to a smaller extent than will a boy, but girls enter many trades and professions, a knowledge of the mathematics of citizenship is a necessity for all, and there still remains the cultural value of the study. The modern world, the problems of economics, sociology, the advances of science and engineering can often only be comprehended in mathematical terms.

In the past, the necessary domestic arithmetic has failed by being artificial, but real life examples can easily be

obtained. Arithmetic may be applied to household accounts, various methods of saving, insurance, banking, etc., methods of purchasing including hire-purchase, gas and electricity supplies, and the most profitable ways of buying them. Mensuration will include work on sizes of rooms, gardens, carpets, curtains and furniture, convenient methods of arranging furniture, lengths of material required for certain purposes and the most economical ways of utilizing them. Many of the weekly and monthly periodicals devoted to the home, needlecraft, dressmaking, etc., contain sets of data which may readily be converted into useful problems. An intelligent observant girl will find many problems, seek methods of calculation and finally interpret the results. As in other arithmetical work 'false accuracy' should be avoided and the importance of rough estimates and checks should be realized. Formerly, problems such as those about papering the walls of a room or on the curtaining of windows were worked to a degree of accuracy which was not only impracticable but also made no allowances for the correct placing of the pattern, wastage, the amount to be allowed for hems and the 'hang' of the curtain.

'This requires a confident use of numbers in a real context, very simple calculation and the power to visualize a situation involving simple spatial relations. It is probable that most problems of this kind calling for estimation and judgment rather than calculation and exactitude can be most effectively treated outside the mathematical lesson. The bearing of the mathematical course on this problem is more indirect than direct since it calls for general mathematical powers but only very elementary mathematical processes.'¹

No general rules for the division of Central School children into A, B and C sets with regard to mathematical

¹B. of E. Pamphlet No. 101.

ability can be given. An ideal system would allow for schemes of work for many more degrees of ability. 'A' children, especially in the upper classes, can be expected to appreciate some of the simpler generalizations of mathematics, to show originality, invention and a logical outlook. With B children the basic arithmetic syllabus will be covered, mensuration can be extended to include surveying and other practical work and the mathematics of citizenship should figure in work which will have a practical keynote. The attainments of C children vary from school to school. Often the arithmetic course will be limited to the calculations of everyday life. Such children usually find their salvation in practical mensuration in the school garden or workshop and in the domestic-science centre. Long calculations with large sums should be avoided, and indeed, there is little to justify such work with children of any grade. Frequent consideration of the child's progress with a view to regrading is necessary. In this connection it is necessary to remark that in districts where the standard of the transfer examination is high, it is possible that the child's apparent attainments will be artificial, mechanical and devoid of intelligent application; and thus a period of consolidation in the post-primary school must be allowed.

Non-selective Central Schools and 'senior-classes' have to contend with the problem of the backward or even mentally-defective pupil. Sometimes backwardness in children rather below average intelligence is due to faults in school organization rather than to physical causes. The child is often promoted either too slowly or too quickly, owing to inefficient methods of testing and grading. Sometimes the change is made needlessly abrupt; there should be a continuity of methods throughout all stages of the school life and the 'spiral method' of teaching arithmetic should embrace the complete school career. However, according to Dr. Cyril Burt, inefficient teaching and school organization

account for less than five per cent. of the cases of backward children, and even this figure is growing smaller.

Every backward child or mentally defective is a complex case to himself owing to the number of hereditary, glandular, sensory and general physical and mental factors which contribute towards the condition. Moreover, few children owe their poor intelligence to a single cause. Specific modes of approach can only be found by careful and often lengthy observation of the child and with full co-operation between the medical officer, psychologist and teacher. The opinions of parents are worth considering, but are often unreliable.

The arithmetic syllabus is here reduced to the bare essentials, and indeed, the teacher has often every reason to be satisfied, if at the age of fourteen the child has grasped the simple arithmetic of everyday life to a standard no higher than that of the normal child at the Primary School stage. Practical routine work such as counting the school milk bottles, checking attendances, reading the rain-gauge, counting eggs, has often been suggested for backward children, but the poorer intelligences bordering on definite mental defectives show such failure of memory, lack of attention and awkwardness of movement that even such tasks are too difficult for them. There is always a lack of co-ordination of motor activities in such children, and the type of 'arithmetic' which they do will aim at helping to overcome these troubles in some measure. Exercises should be given which demand rhythmic movements in counting, which give opportunities for giving greater refinement to the sense of touch and muscular activities, and in general, which link up visual, auditory, tactile and muscular sensations—testing one sensation against another.

The writing of figures should not be introduced too early for this may detract from learning to count. Backward and mentally-defective children usually find great difficulty in making figures and they should learn the shapes by the

Montessori or similar methods. Sometimes a principal cause (though rarely the only one) of backwardness may be traced to a particular sensory defect, in which case the teacher will see that special means are used to employ the other senses, and to obtain the necessary co-ordination. In the case of children who are backward without any mental defect, the work should be very simple, always practical, and wherever possible performed in the open air.

Work indoors may be founded on rhythmic activities, drawing, modelling, paper-cutting using patterns and bright colours; or taking to pieces simple 'structures' and endeavouring to rebuild them again. Finger-counting is permissible but, if possible, work should proceed to simple addition, subtraction and multiplication by means of pattern number-cards, counters, the abacus, coins and picture cards of boys (with two legs each), animals (each having four legs), flowers (each with a given number of petals), etc. Simple counting and picture games will also prove useful. What has been said in Chapter II with regard to arithmetic at the Infant and Nursery School stage applies more forcibly, and with a need for even greater psychological insight, to backward and M.D. children.

The use of Problems in Senior School Mathematics

Problems should be worded in simple language, and it should be realized that many children cannot do the problems because they do not understand their meaning.

It is absolutely futile for a teacher to explain the problem to the child either by hints or blackboard work and then leave it to perform the simple arithmetic. If the problem is too difficult for the child a much simpler one should be given, for it is essential that the child should be able to understand it by reading it through once or twice, and should supply the clue to its solution himself. In the past, and even at the present, many teachers have said that B and C

children in Primary Schools should never be asked to do problems. This arises as the result of taking mechanical arithmetic to a stage where the child can no longer apply it. Mechanical arithmetic only justifies itself when it can be applied to real problems. Even the simplest of processes and the smallest of numbers will find enormous application in the ordinary things of daily life. Often, problems have been graded only with respect to the difficulty of the mathematical processes which they contained. In the 'New Freedom' series an attempt has been made not only to keep the numbers and processes well within the capacity of the child and to find real life problems, but to grade the problems according to the difficulty of the ideas and experiences which they demand.

Testing the Attainments of Children

Teachers will give periodic tests in arithmetic

(a) to check their own work, (b) to find particular difficulties, (c) to grade the children, (d) to find the background and attainments of children who have just entered the school or class.

Following methods which have been employed by the inspectors of the Board of Education, the tests may be divided into Mechanical, 'Mental' (including oral arithmetic, and simple questions where the answers are quickly written down) and Miscellaneous Arithmetic (simple problems).¹ The graded revision exercises in the 'New Freedom' Series serve excellently for this purpose.

At all stages, 'projects' may be introduced to show the application of a 'new tool', to reveal the need of an extension of existing principles and to endeavour to discover it. The number of 'projects' which can be worked is only limited by the time available. In America, the tendency has been to develop the work by 'projects' and heuristic methods to such

¹See Report on the Teaching of Arithmetic, H.M.S.O., 1925.

an extent that a systematic exposition of the groundwork of the subject has sometimes been overlooked. Nevertheless, 'projects' show the real nature of the subject, and they produce a spirit of inquiry and self-reliance. A few projects are given here:

(1) Find the cost of sending a number of postcards, letters, books, parcels, to different parts of the world to investigate the various Post Office rates.

(2) The finances of a small business shop, or café, with special reference to possibilities of gain or loss.

(3) How our school-building was planned.

(4) What is the average consumption of water per head in our district.

(5) How our local rates are assessed and used.

(6) Survey a suitable site for our annual camp.

(7) The finances of a school holiday.

(8) Investigate various statistics with regard to public health, population, employment, trade, the social services, during a period of years and try to interpret them properly.

(9) Examine and measure a tool shed to find the number of 'feet' of timber used in its construction. Estimate the total cost including nails, bolts, creosote, hinges, etc.

(10) Invent certain sets of circumstances with regard to salary, children, health, 'prospects', etc., which might apply to a man of age twenty-one to sixty. Find the income-tax payable or select a suitable assurance policy.

(11) Examine speed records (water, motor-cars, trains, aeroplanes) over a number of years. Draw graphs and try to interpret them.

CHAPTER VI

FRACTIONS, DECIMALS, PERCENTAGES, PROPORTION IN ARITHMETIC AND MENSURATION

Fractions

SIMPLE notions about fractions will arise from the child's experience at home, and it is hardly possible to deal with weights and measures and division without touching upon the matter. 'The idea of a fraction is familiar to children long before they reach the age of seven and there is no reason why they should not be taught early to write down the simple fractions with which they have become familiar in their homes or in their handwork lessons at school.' (Board of Education Report.) However, on examining a large number of seven-year-old children of normal intelligence, it was found that beyond knowing that a thing divides into two halves and four quarters the ideas of fractions were often negligible. Indeed, many of the children had not grasped the idea that the halves were equal, many thought that any small part could be called a quarter (e.g., the 'compartment of an orange') and the ideas of a third and three-quarters were almost unknown. Again, at the transfer examination at the age of eleven, only the very simplest work in fractions is performed and then often not intelligently. It would appear that fractions present a much greater difficulty to children than is generally supposed.

(1) Children tend to take figures at their face value irrespective of their position, e.g., $\frac{1}{7}$ sometimes appears to children to be bigger than $\frac{1}{8}$ because it contains a 'bigger number'.

(2) The idea of multiplying by a fraction causes trouble,

because to a child multiplication should produce a larger number not a smaller one (and the opposite in the case of division).

(3) Children often fail to keep in their minds the nature of the original unit, e.g., if a third is divided into four parts the tendency is to think in quarters not in twelfths, or, as a child put it: 'A quarter of anything is a quarter.'

(4) So many of the seemingly practical questions on fractions which children are given are merely futile.

Early work to explain fractions should always be practical, and even later work should be considerably simplified. Long and artificial complex fractions which usually reduce finally to some simple answer are entirely useless from the practical point of view. Their only justification is that they yield mechanical practice. We may start by cutting up apples or cakes or making models in clay, plasticine or wax, and cutting these up with knives into fractional parts. After this, simple experiments will be undertaken with a foot rule by drawing lines, oblongs and squares and dividing them appropriately. Squares and circles, divided by lines to illustrate simple fractions may be drawn in colours on brown paper, and hung on the walls of a room; or several foot-squares (or circles of about the same diameter) of thin wood may be cut up to show the simple fractions and the relations they bear to one another. Montessori apparatus consists of circles divided into sectors. The names 'denominator' and 'numerator' should be learnt as soon as possible; the former tells us the kind or denomination of the fraction (compare it with other uses of this word) and the latter tells us the number of these denominations. The idea of a fraction can be developed by considering:

(1) Simple objects such as cakes or apples.

(2) A foot rule, where the inch gives a convenient unit. Twelfths and halves, thirds, quarters, sixths, can all be obtained and compared. As has been hinted before, it is

necessary to stress and return to the nature of the whole (or unit) which is being used in early work with fractions.

(3) Simple fractions with money: (a) fractions of 1d., (b) of a shilling, (c) of a pound.

(4) Time: (a) hours, (b) days, weeks, months and years, (c) minutes.

The fractional divisions of time in music—breve, semi-breve, minim, crotchet, quaver, semiquaver and dotted notes, the various time signatures and what they imply, etc., can be represented diagrammatically by pictures of these notes, and also by singing or playing them, preferably on a pipe or some wind instrument. Here the teachers of arithmetic and music may collaborate with good results. During the lessons on fractions a little may be said about the history of fractions. This will not only prove interesting to the children, but the pitfalls of the subject may become more apparent to the teacher when the struggles of the ancients are grasped. They found such difficulties with simple fractions that they tried to avoid dealing with changes in the numerator and the denominator at the same time. The Ahmes papyrus (written before 1700 B.C. and founded on a work said to go back to 3400 B.C.) tells us some interesting things about the manner in which fractions were used by the Egyptians. In common with the Greeks they kept the numerators constant (unity) and varied the denominators. On the other hand, the Babylonians and Romans kept the denominators constant (60 and 12 respectively). Our modern percentage system may be regarded as keeping a constant denominator of 100. Diophantus, one of the last of the second Alexandrian School of great Greek mathematicians (about A.D. 250), had an epitaph which tells us that he passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth and $\frac{1}{7}$ more as a bachelor, five years after his marriage was born a son who died four years before his father, at $\frac{1}{2}$ his father's age. Thus, he died at the age of 84.

If the early practical work has been thoroughly performed, children will see that special means have to be employed in carrying out the simple operations of addition, subtraction, multiplication and division of fractions. Although half-crowns, sixpences, and shillings are all silver coins, one cannot add 3 half-crowns, 5 sixpences and 2 shillings as they stand to make 10 coins of the same kind or denomination. We have to use some means to convert the money to 'something' that will divide into all these things, before we can begin to add them. Thus, we may add them after conversion to sixpences. The idea of Least Common Multiple will grow up through an endeavour to find means of adding fractions. That is the only use for L.C.M. about which we need trouble, and it should then be introduced as a necessary tool. There is no real need to deal with Highest Common Factor. We cannot add $\frac{1}{2}$ and $\frac{1}{4}$ without converting them to $\frac{1}{2}$ s. Then they are of the same denomination and may be added. Problems should be carefully graded so that, at the start, the common multiple is also the *least* common multiple. Wherever difficulties arise the child should at once go back to a practical investigation by drawing lines, or examining his foot-rule. Unintelligent work with simple fractions will show itself in a confusion of the processes used in addition and in multiplication. When more complex work is taken, analogies with simpler cases will often serve to indicate the correct operation which has to be performed. This is particularly the case when cancelling is done. Many teachers have found such difficulties with this process that they insist that it shall be deferred to the last line wherever possible. Sometimes when a numerator or denominator consists of two or more terms connected together by + or - signs the child is tempted to cancel only one of these terms; again, when a number is cancelled by an equal number the result is frequently put down as 0 instead of 1. If any work with complex fractions is under-

taken it should be insisted that the whole of the fraction is brought down at each stage even though this involves some repetition of figures; otherwise, parts of the fraction which do not need immediate attention are sometimes overlooked and left 'high and dry'.

The methods of multiplying and dividing fractions are very easy to learn mechanically but rather difficult to grasp intelligently. The following points may be kept in mind:

(1) Work will commence with the multiplication and division of fractions by simple integers.

(2) The principle may be demonstrated (*a*) by drawing rectangles with sides of 'fractional' length and comparing the areas with that of a square unit, (*b*) by continued addition of a fraction (of an inch) measured out along a straight line.

(3) Division may be illustrated as a process of continued subtraction at the commencement. Later the work may be extended by analogy with arithmetical processes with simple integral numbers.

(4) The equivalence of the word 'of' and ' $\times$ ' should be shown by simple examples.

(5) The child should be encouraged to refer to his foot-rule when difficulties arise. If the processes are learnt and applied, their full meaning will appear when later work is undertaken.

Decimals

When we talk about decimals in schools we usually mean decimal fractions, or numbers which have been extended to the right from integral numbers, and are separated from the latter by a decimal point. Our usual way of setting down any integral numbers is part of our decimal notation, and is the result of counting in tens because we have ten fingers and ten toes.

A systematic treatment of decimal fractions was first given by Simon Stevin (1548-1620) who lived in Bruges, and is

well known for his work on the theory of the inclined-plane. He was able to extend decimal-notation to the right past the units digit. Instead of a decimal point he used 0 and his notation was rather clumsy, though it was interesting in that it indicated the powers of the 10 in the denominator, e.g., he would express 7.845 as $7(0)8(1)4(2)5(3)$ (the 5 in the decimal fraction is $\frac{5}{10^5}$ or $\frac{5}{10^5}$ and so on).

The decimal notation as applied to decimal fractions is worth stressing, for in the words of Cajori, 'the miraculous powers of modern calculation are due to three inventions: the Arabic Notation, Decimal Fractions and Logarithms'.

Two hundred years after Stevin's treatment of decimal fractions the Metric System was developed and used after the French Revolution. A system of interrelated weights and measures, founded on decimal subdivision and easily understood by all intelligent people (and not retained as a secret code by a few wealthy merchants who could use it for cheating the poor), naturally commended itself to the post-revolution commission. (Here is an excellent topic for the Senior School.)

Early work in decimals will involve:

(1) *Drawing and measuring with ruler in inches and tenths of an inch.* Teachers should be able to deal simply and practically with the metre scale, and its subdivisions: deci-, centi-, and millimetres, in the Primary School; but usually a thorough treatment of the Metric System will be taken more conveniently in the Senior School. Any work which is done on this in the Primary School should confine itself to a knowledge of comparative and actual sizes of the various units considered, e.g., 10 cms. is almost 4 inches. How long is my little finger? How far can I span between first finger and thumb?—between little finger and thumb? What is the length of my arm, the distance between finger-ends with arms outstretched, my normal pace? How thick is a match stick, a cigarette card (by using fifty), my Scout stave? What

are the most convenient units to employ for these measurements? How did men manage to measure in past days—e.g., as recorded in the Old Testament?

(2) *Learning to talk about decimals and to read numbers in decimal notation.* Practice in this may be had by using six, seven, or eight vertical columns for Thousands, Hundreds, Tens, Units, Tenths, Hundredths, Thousandths, etc.

Thousands	Hundreds	Tens	Units	Tenths	Hundredths	Thousandths	Ten Thousandths
6	7 6	8 7 6	0 8 7 6	 ·8 ·7 ·6 ·0	 8 7 6	 8 7	 8

Children should be given constant practice in reading the numbers, e.g., sixty-seven point eight, and point nought six seven eight, etc. When 0 occurs in the decimal fraction it should always be read as 'nought'. Children may be led to see that here is a simple and useful extension of the ordinary integral numbers which we have used hitherto. As facility is gained in expressing decimal-notation in words, and words in decimal-notation, children should be able to read numbers of seven digits to the left of the decimal point and six places to the right. In England we rightly insist that the decimal point should be placed above the line at about the middle of the figures, and *not* on the line where it might be confused with a comma. Some foreign mathematics books print the integral number on the left of the decimal point in heavier

type. The equivalence of simple decimal and vulgar fractions can be demonstrated by reference to the columns which we have used for tenths, hundredths, etc. Each way of representing a number presents definite advantages under certain circumstances; and one of the most important points which should appear in the teaching of mathematics at every stage is the selection of the most convenient forms of expression and manipulation for the particular task which is presented. The old method of converting a decimal fraction to a vulgar fraction by putting it over the number made by putting 1 for the decimal point and a nought for each figure in the decimal must be justified, and used only for the fractional part of a decimal. The idea of *significant* figures should be mentioned in the Primary School and stressed again later. The reverse process of converting a vulgar fraction into a decimal is a little more difficult to explain, but it can be approached through simple examples like $\frac{1}{10}$, $\frac{1}{2}$, $\frac{1}{100}$, $\frac{1}{4}$, $\frac{3}{4}$, etc. The process of dividing denominator into numerator is really equivalent to arranging the fraction so that its denominator is some power of ten, and then dividing by this power of ten by moving the decimal point.

$$\frac{1}{4} = \frac{25}{100} = .25$$

Hence, to prepare properly for this operation a revision of vulgar fractions, an idea of 'powers of 10', and the moving of the decimal point to the right or left when multiplying or dividing respectively by powers of 10 will be necessary. The idea of an index notation is so important that it should be approached through powers of two and ten as soon as possible, though many teachers would wish to defer it to the late Senior School stage. School work in decimals in the past has relied to a large extent on remembering 'rules'. Rules have their place in school mathematics, but they should be reduced to the absolute minimum and always safeguarded

by an intelligent knowledge of how they are obtained and when to use them. In the case of work in decimals, common-sense and rough estimates always will suffice to show whether an operation is correct or not.

If exercises are carefully graduated, and there has been sufficient early drill in setting down decimal fractions in columns there should be no difficulty in performing simple addition and subtraction. The best ways of performing the multiplication and division of decimals have been the subject of some controversy. As we have indicated before, teachers should be prepared to confer with one another so that there is uniformity in the basic operations of early arithmetic, in order that the child may pass smoothly from one part of his school life to the next. There is now considerable collaboration between the teachers of various subjects in Senior Schools, but this is not sufficient. There should be an understanding by teachers at all stages, of the work and methods of the schools for younger and older children. (In a more general way, there is still need for the further integration of the various aspects of educational activity and influences.)

Operations with decimals should be of the simplest nature in Primary Schools, but most of the work will be undertaken in Senior Schools.

Standard Form. A number is said to be in standard form when it lies between 1 and 10. In dividing or multiplying decimals the divisor or multiplier is arranged in standard form.

Consider 7894.27×36.8 .

To arrange this in standard-form move the decimal point one place to the left in the number on the right (multiplier). This is equivalent to dividing the multiplier by 10. To counteract the effect of this on the final result we therefore have to multiply the number on the left by 10. This is done by moving the decimal point to the right, i.e., the expression

becomes 78942.7×3.68 . The common-sense aspect of this should be insisted upon, but the rule: 'in multiplying move the decimal points the same number of places in opposite directions,' may be remembered.

When dividing it is clear that in order not to upset the quotient we must treat dividend and divisor alike, and thus 'we must move the decimal points the same number of places in the same direction'.

The advocates of standard-form claim that it allows rough estimates to be made and produces systematic work especially when contracted methods are used. On the other hand, rough checks may be made without standard-form, and if the processes are performed intelligently it is not difficult to place the decimal point correctly without using this method.

In all work with decimals the answer should be safeguarded by making rough checks, and in later work stress should be placed on contracted methods, approximate answers and results correct to a given number of places of decimals. The multiplication and division of decimals very often leads to unintelligent work; for instance, if 23.87 is multiplied by 104.82 and the result is required to be accurate to the first decimal place, it is not satisfactory to work to four decimal places and then remove three of them. Suppose now that the number 23.87 is given accurate to the second decimal place, instead of being *exactly* 23.87 . It is clear that it can contain errors up to $.005$ or $\frac{1}{200}$ of a unit. Thus, when multiplied by 104.82 (which is approximately 100) the possible error of $\frac{1}{200}$ is magnified to $\frac{1}{2}$ or $.5$. Thus, in the final result it will only be safe to give the answer to the nearest integer. There are equivalent aspects of this in division, where if the divisor is greater than 1 the error will be reduced, and if it is less than 1 it will be increased. A failure to grasp the nature of approximation and accuracy to a specified number of decimal places often leads to absurd 'false accuracy' when working out problems such as those

involving the decimalization of money and of quantity, and thereafter finding the price of the given quantity by multiplying the two decimals.

In actual circumstances, in science and engineering and indeed in nearly all the applications of mathematics, 'things do not work out exactly.' The number has to be expressed approximately, the maximum error to which it is liable should be known, and the subsequent history of this error (which may become negligible or on the other hand may assume considerable proportions) must be traced throughout the whole of the processes. This is a difficult problem, but it permeates practical mathematics to such an extent that it must be kept in mind from the start.

Recurring or Circulating Decimals

Most teachers will probably omit all work on recurring decimals, and indeed it is difficult to justify their inclusion, as fractions are usually a more convenient form of expression; and if the decimal form is really necessary it can be used to the desired number of decimal places without reference to the 'circulation'. Recurring decimals are best expressed in terms of infinite geometric series, which is clearly beyond the scope of Central School work. However, some teachers may like to give the rule for converting a recurring decimal to a fraction by putting the fractional part of the decimal over a denominator made by writing a 9 for every recurring figure.

Percentages

A few examples of the addition of fractions requiring the determination of a common denominator, will suffice to show the simplicity of a system of fractions where there is the standard common denominator of 100 always implied.

A percentage may be introduced by showing how we may

write a fraction with a denominator of 100 in another way

$$\frac{31}{100} = 31 / 100 = 31\%$$

(The sign % being a neat way of writing $/100$.)

This at once gives the clue for converting percentages into fractions, and the reverse process will be seen to be equivalent to changing the denominator of the given fraction to one of 100, and implying this in the % sign, e.g.—

$$\frac{4}{5} = \frac{400}{500} = \frac{400}{5}\% = 80\%$$

$$\text{or more briefly, } \frac{4}{5} = \frac{4}{5} \times 100\%.$$

Here practice with many quick oral examples will prove useful. The ease with which percentages can be converted to decimal fractions and vice versa will also be noted.

The use of the term 'a hundred per cent.' meaning 'the whole thing', 'unity', or 'entirety' is so common that it may prove a starting-point for a consideration of percentages without the use of fractions. However, as in the case of a fraction, when a percentage is mentioned, some practical meaning should be applied to it. For instance, I am given the result of a scientific experiment and told that it is correct to within 5%. Suppose the answer I am given is 500 feet. Therefore I believe that if the correct result could be determined it would lie between 475 ft. and 525 ft. A graph or a model will often make these matters plain to children.

Oral drill will be useful in fixing the conception of 100% as 'the whole' or the 'original', and if this has been done there will be little trouble with subsequent work on percentage increases, gains or losses, e.g., if 17% of the children taking an examination failed, what percentage passed? This is followed by examples of gains and losses, expressed as a

percentage, leading to more difficult problems, e.g., the population of a town in 1901 was 56,200. In 1911 it was 62,500. What is the percentage increase of the population during the period of 10 years?

$$\text{Increase in number} = 6,300$$

$$\text{Fractional increase on original number} = \frac{6,300}{56,200}$$

$$\text{Percentage increase} = \frac{63}{562} \times 100\%$$

etc.

(Cancel and divide; leave the arithmetical working to the last.)

In expressing one quantity as a percentage of another, several devices may be employed, the quickest and most convenient being chosen. Sometimes a simple 'practice' method may be useful, and closely related to it a method founded on aliquot parts. (See section on Money.)

Example: A boy weighed 7 stones, $4\frac{1}{2}$ lbs. in 1935 (Jan. 1st) and 8 stones, $2\frac{1}{2}$ lbs. in 1936 (Jan. 1st). Find his percentage increase in weight.

(A convenient unit will be $\frac{1}{2}$ lb. though largest possible unit should be used.)

$$\text{Increase in weight of boy} = 12 \text{ lbs.}$$

$$\text{Fractional increase in weight of boy} = \frac{\text{increase in weight}}{\text{original weight}}$$

$$= \frac{24}{205}$$

$$\% \text{ increase} = \frac{24}{205} \times 100\%$$

etc.

(Cancel and divide.)

Profit and Loss

Profit or Loss may be reckoned in several ways:

(1) As a sum of money obtained by subtracting cost price from selling price, or vice versa in the case of a loss.

- (2) As a percentage of the cost price.
- (3) As a percentage of the selling price.
- (4) More rarely, as a percentage of the sum of the cost and selling prices. (*N.B.* Selling price is sometimes called 'turnover', a term which is also occasionally applied to the sum of cost and selling prices.)

To reckon profit or loss as a percentage of the cost price is logical and brings the matter into line with other increases or decreases of an original quantity, e.g., an increase or decrease in population. Shopkeepers, however, usually reckon profit as a percentage of the selling price as it is more convenient owing to their method of keeping accounts, and also it appears as a more modest rate of profit.

Work may be commenced as follows:—

- (1) Given cost price and percentage profit or loss on cost price, to calculate selling price.

e.g., if profit is 15 %

Selling price (S.P.) = 115 % of cost price (C.P.)

or if loss is 7 %

Selling Price = 93 % of Cost Price.

- (2) Given cost price and selling price to calculate percentage profit on cost price

$$\text{Profit} = \text{S.P.} - \text{C.P.}$$

$$\text{Profit as a fraction of C.P.} = \frac{\text{S.P.} - \text{C.P.}}{\text{C.P.}}$$

$$\text{Profit as percentage of C.P.} = \frac{100 (\text{S.P.} - \text{C.P.})}{\text{C.P.}} \%$$

- (3) Given cost price and selling price to calculate percentage profit on selling price.

$$\text{Profit as percentage of selling price} = \frac{100 (\text{S.P.} - \text{C.P.})}{\text{S.P.}} \%$$

Many profit and loss problems may be worked by using the methods of proportion, considering at the commence-

ment that the cost price is £100, and reckoning proportionally. This is open to the objection that the cost price is usually *not* £100, and such a method is not necessary if the meaning of percentage has been thoroughly grasped, e.g.

A man loses 10% by selling a house for £720. For what should he have sold it to have gained 15%?

(£720 is 90% of the cost price.*)

$$\therefore \text{Cost Price} = \frac{10}{9} \times £720.$$

(Second selling price = 115% of the Cost Price.)

$$\therefore \text{required selling price} = \frac{115}{100} \times \frac{10}{9} \times £720 \quad \text{etc.}$$

(*Here the temptation is to take 10% of £720 and add it to this to find the Cost Price.)

Unless the cost price is explicitly required, all working should be left to the end of the problem.

Other typical problems concern the various 'percentages' made by the manufacturers, agents, wholesalers, and retailers of goods. In these cases it is necessary to ascertain on what price the 'percentage' is allowed.

Example: An article is sold for 10 guineas. The retailer makes 30% profit, the wholesaler 15%, the agent 10%, the manufacturer 20%. What is the basic cost of manufacture?

A problem of this type should be discussed from the practical point of view. In this case the assumption is made that each percentage profit is reckoned on the previous price. *This is not always so.* Let Basic cost price be £x.

$$£x \times \frac{120}{100} \times \frac{110}{100} \times \frac{115}{100} \times \frac{130}{100} = £10\frac{1}{2}$$

$$\therefore £x = \frac{£21}{2} \times \frac{100}{120} \times \frac{100}{110} \times \frac{100}{115} \times \frac{100}{130} \quad \text{etc.}$$

Simple algebraical methods will facilitate the expression

of the problem, but it must be realized that in practice there are many ways of reckoning profits.

Discount

Discount is usually reckoned as a percentage of the selling price, and taken from it.

Example: A tailor marks a suit so as to give a profit of 30% on the cost price. He allows 5% discount on the marked price and finds that he has actually made a profit of £2 on the suit. What was the cost price? (In actual practice the figures might not be so exact.)

$$\text{Marked Selling Price} = \frac{130}{100} \times \text{Cost Price}$$

$$\text{Selling price after discount} = \frac{130}{100} \times \frac{95}{100} \times \text{Cost Price}$$

$$(\text{N.B. not } \frac{125}{100} \text{ Cost Price.})$$

$$\therefore \text{Profit} = \frac{130}{100} \times \frac{95}{100} \text{ C.P.} - \text{C.P.} = £2.$$

$$\therefore \left(\frac{130}{100} \times \frac{95}{100} - 1 \right) \text{C.P.} = £2$$

$$\left(\frac{247}{200} - 1 \right) \text{C.P.} = £2$$

$$\text{C.P.} = £2 \times \frac{200}{47}, \text{ and so on.}$$

In the case of sums of money required as percentages of £1, if a simple conversion to fractions is not obvious, it is sufficient to decimalize the money and multiply by 100 by moving the decimal point two places to the right.

$$18\text{s. } 4\frac{1}{2}\text{d.} = £.919 = 91.9\% \text{ of } £1.$$

(Correct to
3 places
of decimals)

(Correct to
1 place of
decimals)

Ratio and Proportion

Proportion has probably been responsible for many more absurdities than any other branch of arithmetic. In the past so much teaching of this subject has been useless, or even worse than useless, for there has often been a violation of 'common sense' principles. The chief troubles have been:

1. The 'rule of three' was learnt by rote and then applied to the three quantities concerned. If the rule was thoroughly known and the data were supplied in conventional order it was possible to obtain a correct answer. The rule made no provision for an intelligent grasp of the process, and whether it was valid or not to apply it. Happily the rule in its old form would appear to be quite dead.

2. Intelligent work on proportion is quite impossible unless children are sufficiently intellectually mature to understand the idea of fractional multiplication and division, and to realize the logical ideas which we assume when we think of variation of one thing with another.¹ This is often related to ideas of causation and of correlation, and until children have examined and thought about many practical examples and discussed them, it is useless to start a systematic study in mathematical terms.

Work on proportion may well begin by a discussion, which need not be hurried, of a number of examples where proportion does not exist, where it only partially exists, or again where it is not simple proportion. There is no need to give a lengthy list for both teacher and child will obtain amusement in thinking of other cases.

The following examples should be discussed from every angle. Quite difficult ideas of logic are sometimes involved, which will reveal at once both the value and difficulty of an elementary but thorough and intelligent treatment of proportion.

¹See E. M. Renwick, *The Case against Arithmetic*.

1. A cricketer made 10 runs in half an hour. How many would he make in an hour and a half?

2. Sir Malcolm Campbell's car can go 300 miles in an hour. How long would it take to go round the world (25,000 miles)?

3. I can write 3 pages of this book in an hour. How many days, working 12 hours a day, were necessary to write it?

4. A boy ate $\frac{1}{4}$ lb. of chocolate in 10 minutes. How much could he eat in an hour?

5. 50 men built a concrete dam in 160 days. How many would be required to build it in 2 days?

6. A man can see 10 miles out to sea from a cliff 100 ft. high. How far can he see from a cliff 200 ft. high?

Science provides us with many examples of proportion which is direct, or 'linear'. Many of the difficulties in simple quantitative physics are due to a failure to grasp the different types of variation, such as inverse proportion (Boyle's Law), variation with the direct square (electric current and heat losses), the inverse square (e.g., intensity of illumination and distance from source of light, or time of exposure in taking a photograph and the diameter of the aperture of lens), etc. There are also hundreds of cases where the variation is not given by any simple expression such as a square or an inverse square. For instance, the wind resistance against a moving vehicle is taken as being directly proportional to its speed for a small velocity, but as this increases, the 'law' becomes very complex, and ultimately the wind resistance is proportional to a large power of the speed. Clearly this is difficult work, but the intelligent student will think about it later, as it has tremendous significance in engineering and the applied sciences.

Sufficient has been said to show that work in proportion could be made a useful arithmetical tool if it were generally simplified, confined to practical examples; and even in these

cases, a discussion allowed of every little pitfall or absurdity to which the problem may lead.

Proportion must make an appeal to the intelligence; in the past it has failed to do this in a large measure. In the teaching of proportion, as in every topic in mathematics, it is impossible to deal with the subject, exhaust it and pass on to the next topic. Constant reference must be made to subjects which have been commenced at an earlier date, and for which the time has become ripe for reconsideration. Proportion calls for an intelligent grasp of multiplication and division of fractions, which at an earlier stage are the cause of some difficulties. The 'unitary method' has often been taught, as it is easy to understand and makes little demands on the intelligence of the 'weaker brethren'; but it is often long and tedious and leads to the grossest absurdities of statement, as almost any example worked by this method will show. There is also the serious objection that it is quite possible to work a number of examples by the unitary method and yet fail to grasp the essential nature of proportion. Nevertheless, a partial unitary method is often useful as a start, if examples are chosen wisely.

Formal work may be commenced by considering proportions arising from multiples, and fractions with a numerator of one. This will be continued by problems involving the use of simple fractions (e.g., I can buy 4 books for 12 shillings, what do 3 cost?) and continued to include more difficult fractions.

$8\frac{3}{4}$ ounces of silver cost 18s. 11½d., what would be the cost of 10 ounces?

The child asks:

- (1) In what units must I give the answer?
- (2) Will it be greater or less than the quantity in the question?
- (3) What is the fraction I must use for multiplication?

Algebra and simple graph-drawing will later allow the work on proportion to be extended and generalized.

Even simple proportion may be set out in 'the x ' or equation form.

$$\frac{x}{a} = \frac{b}{c} \quad \therefore x = \frac{a \times b}{c}$$

The old 'dot' method of setting out proportion $x : a :: b : c$ is superfluous and ought to be abandoned.

Inverse Proportion

An example will illustrate this: A unit of electricity is consumed when a '60-watt' bulb is 'burning' for 16 hours. In how many hours will a '25-watt bulb' consume a unit? An algebraic treatment of the matter is useful and this topic has been more fully treated in the section on graph-drawing. In this case two things vary so that their product is a constant. An example which appeals to children is 'the length and breadth of a rectangle of constant area'

$$l \times b = k \text{ (constant)}$$

$$l = \frac{k}{b} \quad b = \frac{k}{l} \text{ (more breadth, less length)}$$

$$\text{ (more length, less breadth)}$$

which is sometimes written $l \propto \frac{1}{b}$ $b \propto \frac{1}{l}$

Compound Proportion

Here many of the old examples which occupy a large number of lines of print and contain much data, could be replaced by practical examples arising from surveying, mensuration and science. All later work in arithmetical proportion should develop side by side with its algebraic and geometrical equivalent topics. In working compound proportion sums the following points may be kept in view:

(1) On reading the problem through once, do not try to fix all the data in the mind. Find out what is required (days, men, money, etc.).

(2) From the data given frame a statement which ends with the thing required.

(3) Deal with each proportional change separately. Arrange neatly with arithmetical statements on the right.

(4) Obtain the final shape of the answer before beginning to cancel the fractions.

(5) Check separately: (a) the stages of the argument (is it more? or less? etc.), (b) the arithmetical working.

Sometimes, problems on compound proportion may be reduced to simple proportion before commencing the working: e.g., 'Five days at six hours a day' may be thought of as 30 hours.

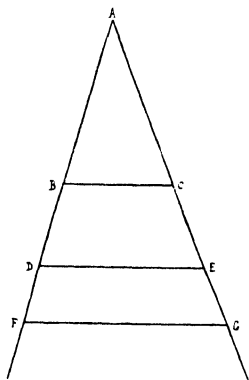
Similar Figures

The principle of the similarity of figures is of exceptional importance. Its simplicity and wideness of application has often been obscured in the past by approaching it through the theorems of Euclid's sixth book, which are very dull in themselves, come too late in a course of deductive geometry founded on Euclid's sequences, and are not always satisfactory from the point of view of strict logic. The idea of a figure having the same shape as another, but differing from it in size is one which sooner or later dawns upon the mind in these days of cheap maps, scale models, 'snap-shots' and enlargements, cinema films and lantern slides and other similar things. By considering a photograph or a model locomotive a child may be led to see that in similar figures or objects the ratios of equivalent measurements must be preserved. The eye is often quite sensitive in detecting small departures from true similarity. Thus, if the length of a boy's nose is a quarter of the distance between his chin and the top of his forehead, he would probably be unrecognizable if a photograph distorted it so that its length became a third of this distance on the picture. We may now pass on to a consideration of the similarity of simple

geometrical figures. All squares, circles and equilateral triangles are readily seen to be similar, but it must also be grasped that the diagonals of the squares are proportional to the sides, the perpendiculars from the vertices to bases of the equilateral triangles are also in the ratio of the sides, and so on for any corresponding lines which may be drawn in similar figures.

The next step is to deal with similar triangles. This should be approached experimentally by drawing and measurement.

Two straight lines are drawn from the point A and a triangle ABC is completed. DE and FG are drawn parallel to BC. By measurement it is found that:



$$\text{Angle } ABC = \text{angle } ADE = \text{angle } AFG$$

$$\text{Angle } ACB = \text{angle } AED = \text{angle } AGF$$

$$\text{Also, } \frac{AB}{AC} = \frac{AD}{AE} = \frac{AF}{AG} \quad \text{and so on.}$$

If triangles ABC, ADE and AFG are cut from separate sheets of cardboard or plywood, the experiments will be facilitated. It should also be shown that other figures which are equiangular are not necessarily similar (rectangles

are equiangular but not usually similar), but on the other hand similar rectilineal figures are equiangular.

These are important ideas for they find many applications in mensuration, surveying and map-making, art and drawing, physics, mechanics and engineering; and the mathematical treatment should be combined with scale-drawing, problems arising from other school subjects, or the child's hobbies, such as photography, model-making, scouting, etc.

Methods of drawing figures similar to given figures are sometimes useful. By taking a convenient point in the given figure and drawing lines, the given figure may be split into a number of triangles. By drawing lines parallel to the sides of the given figure, a number of triangles similar to those composing the given figure will be obtained.

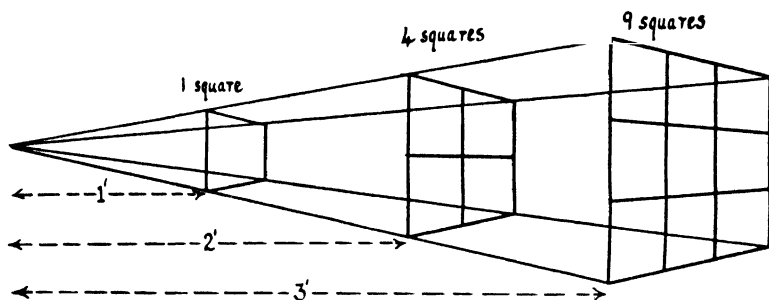
Similar Figures and their Areas

Starting with squares, it is easily seen that their areas are proportional to the squares of their sides. If the simple mensuration of the circle has already been taken, it readily follows that the areas of circles are proportional to squares of their diameters (or radii). Difficulties sometimes begin to arise when we endeavour to show that the same principle applies to triangles and indeed to any plane figure. In the case of a triangle its area is equal to $\frac{1}{2}$ Base $\times$ Perpendicular height. Now if the base is increased so that its new length is $x \times$ old length (i.e., in the proportion $\frac{x}{1}$), the perpendicular will be changed in the same proportion if similarity is to be preserved.

$$\begin{aligned}\text{Thus new area} &= \frac{1}{2} \text{ Base} \times x \times \text{Perpendicular height} \times x \\ &= x^2 \left(\frac{1}{2} \text{ Base} \times \text{Perpendicular height} \right) \\ &= x^2 \times \text{original area.}\end{aligned}$$

The principle may be extended to include any plane figures and many other surfaces (such as the curved surfaces of cones and spheres). The latter case may be developed to

show the law of inverse squares which finds such important applications in light (intensity of illumination, etc.), magnetism, electricity and indeed in any form of radiation.



A wire model as above, and practical experiments with an improvised shadow or grease-spot photometer and numbers of candles, will also serve to demonstrate the principle of the law of inverse squares.

In similar figures, as areas are proportional to the squares of corresponding lines in the figures, it follows that the lengths of corresponding lines are proportional to the square roots of the areas of the similar figures. An example will show the principle in use.

A wool mat 5 ft. by 3 ft. contains 9 lbs. of wool. What will be the dimensions of a mat of similar shape requiring 6 lbs. of wool? (the thickness of wool being constant).

The areas of the two mats are in the ratio $\frac{6}{9}$ or $\frac{2}{3}$

$\therefore$ Corresponding sides are in the ratio $\frac{\sqrt{2}}{\sqrt{3}}$

$\therefore$ The lengths of the sides are

$\sqrt{\frac{2}{3}} \times 5$ ft. and $\sqrt{\frac{2}{3}} \times 3$ ft. respectively.

The Volumes of Similar Solids

'The volumes of similar solids are proportional to the cubes of corresponding dimensions.' This fact is easily seen in the case of cubes, but it is less obvious when other figures are considered. The difficulty may be overcome by experiment, and by considering successively rectangular solids, pyramids, cones and spheres, and then more complex figures such as are found in scale-models. A gallon measure similar in shape to a pint pot is found to have its linear dimensions just twice those of the latter. This principle finds many applications, and it is due to a failure to grasp the fact that volume increases with the cube of a linear dimension, which leads to the tendency to under-estimate the capacities of fairly large vessels. The gallon measure mentioned above looks surprisingly small! Lack of space precludes a long discussion of the applications of these simple principles of proportionality, but a few may be mentioned.

(a) Volumes are proportional to the cube of linear dimensions and the surface area is proportional to the square of linear dimensions.

E.g., eight small equal raindrops condense into one of double the diameter of each of the original drops, and their total surface area is thus reduced to a half. If the drops originally held a surface charge of electricity, this is now distributed over a surface half as large as before. The voltage is therefore doubled. As the process of condensation proceeds the voltages increase to such an extent that the result is an electrical discharge (lightning). This type of example finds many applications in the sciences.

There are many further examples from nature study, e.g., the larger the animal or structure, the thicker must be its legs or supports, in more than linear proportion to its height; for weight increases with the cube of a linear dimension but the strength of a limb or rod does not. This

finds interesting applications in the sizes and structure of animals. (See *Living Machinery*, by Hill.)

(b) In making a model of an engine, the heating surface (e.g., firebox area) is reduced by the square of the fraction of the scale, but the volumes of the boilers and the cylinders are reduced by the cube of the fraction of the scale, e.g.,

A Model with linear dimensions $\frac{1}{20}$ th of original

Firebox area $\frac{1}{400}$ of original

Capacity of boilers, cylinders, etc. $\frac{1}{8000}$ of original.

The principle may be applied (approximately) to many models, sometimes including electrical machinery. Models of aeroplanes and hulls of ships are often made with a view to predicting the performance of the full-sized structure. Although this is definitely well above the standard of the Central School, it should be kept in mind for future work, and the uses and dangers of calculating from models noted. A scale model of a pendulum clock will not keep time. Why? There are many further examples of a simpler nature:

Example: What should be the proportionate diameters of $1\frac{1}{2}$ d. and 1d. oranges in order that they shall be equal value for money? (subject to certain assumptions). As the volumes should be proportional to the prices, i.e., 3 to 2, the ratio of the linear dimensions should be the cube root of this: $\sqrt[3]{\frac{3}{2}}$ to 1 or 1.145 to 1 (or about $\frac{8}{7}$)

Thus, a $1\frac{1}{2}$ d. orange only needs to be $\frac{1}{7}$ th greater in diameter than 1d. orange to be of equal value for money.

(2) A statue is 20 ft. high and is made of solid stone. A scale model of wood is 1 ft. high and weighs 3 lbs. Equal volumes of wood and stone have weights in the proportion of 2 to 7, find the weight of the statue. Suppose the statue were made of wood. Its volume is $(20)^3 \times$ the volume of the model; its weight would be $(20)^3 \times 3$ lbs. if made of wood. But it is made of stone, which is heavier in the proportion of 7 to 2.

$$\begin{aligned}\therefore \text{Weight of statue} &= (20)^3 \times 3 \times \frac{7}{2} \text{ lbs.} \\ &= 6000 \text{ stones} \\ &= 37\frac{1}{2} \text{ tons.}\end{aligned}$$

Proportional Division

Simple examples from cooking, making alloys, mixtures of chemicals and other ingredients for making fertilizers, solutions, paints and other substances used in the applied sciences will show the principle of proportional division. Books on domestic science, agriculture, horticulture, the 'useful arts', painting, handicraft, engineering and chemistry will supply many examples. (At first, questions concerning the volumes of mixtures should be avoided, for shrinkage of the total volume of the ingredients often takes place during solution. Also, problems concerning density arise.)

Example: An alloy contains 5 parts by weight of tin, 12 of lead and 1 of antimony. Find the weight of each ingredient necessary to make a set of organ pipes weighing $1\frac{1}{2}$ cwt. (The alloy given above is commonly used nowadays, but is 'poor metal'.)

Notice that each 'part' may be taken as a unit of weight. We could, for instance, make a sample of the alloy by melting together 5 lbs. of tin, 12 lbs. of lead and 1 lb. of antimony, or $2\frac{1}{2}$ lbs. of tin, 6 lbs. of lead and $\frac{1}{2}$ lb. of antimony, or again, 15 lbs. of tin, 36 lbs. of lead and 3 lbs. of antimony. The total number of 'parts' = $5 + 12 + 1 = 18$.

$\therefore$ Amount of each ingredient expressed as a fraction of whole is

Tin $\frac{5}{18}$, Lead $\frac{12}{18}$, Antimony $\frac{1}{18}$,

Actual quantities in $1\frac{1}{2}$ cwt. of alloy:

$$\text{Tin} \quad \frac{5}{18} \times 168 \text{ lbs.}$$

$$\text{Lead} \quad \frac{12}{18} \times 168 \text{ lbs.}$$

$$\text{Antimony} \quad \frac{1}{18} \times 168 \text{ lbs., and so on.}$$

A simple algebraic method will often prove useful.

The working may be checked by adding the separate quantities.

The proportional division of a line may be shown as an illustration of this process.

CHAPTER VII

MONEY; TIME; THE METRIC SYSTEM

Money

PROBLEMS involving money may commence at the Infant Stage by handling real or cardboard coins to encourage familiarity with their size, shape and appearance. The relative values of the coins may be taught by using the following methods.

(1) Shillings and Pence. By picking out numbers which are multiples of 12 in a '10 square' number-chart, the child is made familiar with the 12 times table, as far as 60 or 72 to begin with.

(2) By using real or cardboard coins simple amounts given in shillings and pence are replaced by pennies, and alternately, a number of pennies are taken and counted out in dozens and replaced by shillings.

(3) By using coloured pictures and charts, or 'playing at shops', the child is given an opportunity to see the cost of simple articles in his own experience, e.g., a book, a doll, 2 ozs. sweets, a ball, etc. He will determine by the use of real or cardboard coins how these simple sums of money may be made up, and how change from sixpence or a shilling might be given to him with various coins.

Various large wall charts may be made to show the subdivisions of a shilling (two sixpences, four three-penny pieces, twelve pence, twenty-four half-pence, forty-eight farthings) and by using analogous methods the pound may be built up from a shilling. Children are always interested in working 'shopping-cards' illustrated with pictures of the articles. Cards may be 'home-made', and deal first with shopping at single shops—greengrocers, grocers, hardware

dealers, Woolworth's, drapers, etc., and later with an 'afternoon's shopping' including visits to various stores. A picture of a street may be supplied and the child encouraged to make a note of the items required under suitable headings, so that the shops may be 'visited' systematically.

Children at the Primary School stage are interested in the history of our and other money systems. In the third century B.C. the Romans first made a silver 'penny' (denarius) (notice d. used for 'penny'). The word pound (libra, contracted to l. or £) is a reference to the balance used for weighing pieces of gold which were made into 72 smaller coins called aurei. Schilling or shilling is of Anglo-Saxon derivation; and the origin of our present word farthing or fourth-*ing* is obvious. History is full of references to coins, such as the well-known Biblical incident, 'Show me a penny,' and Domesday Book mentions a dozen coins including pound, ounce of gold, marc, shilling, penny, half-penny, farthing—even the smallest of which was almost certainly of silver. The stories of the debasement of currencies, the clipping of coinage, coining, the introduction of paper money, legal tender, the costs of commodities at various times in history, etc., are all interesting. Money and financial matters will be discussed again when senior forms are studying the arithmetic of citizenship. Work at the Primary School stage may be varied by a 'play-way' with a full-size or model shop, a 'Bank' and a 'Post Office'. Many children have these and use them at home.

The 'First Four Rules' applied to Money:

Addition

The principle may be illustrated by using real or cardboard coins. The order of approach will be

- (1) The addition of pence requiring conversion.
- (2) The addition of shillings and pence without necessity of conversion, i.e., 5s. 3d. + 6s. 8d.

(3) The addition of shillings and pence, with the necessary conversion.

(4) The above with farthings and halfpence added.

(5) Compound addition. If there are only a few amounts to be added the pence may be converted to shillings as the addition proceeds; but if there are many amounts to be added, the pence may be written down before being converted to shillings; but when the shillings are added the units figure may be written down at once, as it will not be altered by the division by 20.

Subtraction

Methods of (a) Decomposition, (b) Equal Addends and (c) Complementary Addition analogous to those used in the subtraction of ordinary numbers may be employed. As before, the processes may be illustrated by setting the amounts out in coins in boxes divided into compartments for pounds, shillings, pence and fractions of a penny.

Multiplication

Work will be commenced with short multiplication with multipliers not greater than 12. This may be varied and extended by factor multiplication. Numbers such as 32 (8×4), 45 (5×9), etc., are used at the start and may be followed by numbers which do not factorize, e.g.

$$(a) 67 = 11 \times 6 + 1$$

$$(b) 53 = 9 \times 6 - 1$$

In case (a) the given sum of money is multiplied by 11 by the 'short' method, then by 6 and finally the given sum is added.

(b) the given sum is multiplied by 9, then by 6 and the given sum subtracted.

Such exercises are useful for revising the nature of multiplication, and they give practice in finding the easiest method of working a particular problem.

'Long' Methods of Multiplication

The multiplication of awkward sums of money by large multipliers has been a feature of standard arithmetic in the past. Its importance has been overrated and simpler problems and less cumbrous methods will serve for most practical purposes.

(a) The Method of Reduction.

The following example will illustrate this:

$$\begin{array}{r}
 \text{£}2 \text{ 7s. } 8\frac{1}{4}\text{d.} \times 278 \\
 \begin{array}{r}
 \text{£} \quad \text{s.} \quad \text{d.} \\
 2 \quad 7 \quad 8\frac{1}{4} \\
 278
 \end{array} \\
 \begin{array}{r}
 \frac{1}{4}\text{d.} \times 278 \qquad \qquad \qquad 69\frac{1}{2}\text{d.} \\
 8\text{d.} \times 278 \qquad \qquad \qquad 2224 \\
 \hline
 12)2293\frac{1}{2}\text{d.} \\
 \quad 191 \text{ s.} \quad 1\frac{1}{2}\text{d.} \\
 \hline
 7\text{s.} \times 278 \qquad \qquad \qquad 1946 \text{ s.} \\
 \hline
 20)2137 \text{ s.} \quad 17\text{s.} \\
 \quad 106 \text{ £} \\
 \quad 556 \text{ £} \\
 \hline
 662 \text{ £} \quad \text{£}662.
 \end{array}
 \end{array}$$

Ans.: £662 17s. 1½d.

At a later stage some of the working may be done in the margin.

(b) Multiplication by separate digits.

£2 7s. 8½d. by 278. ($278 = 1 \times 8 + 10 \times 7 + 100 \times 2$)
(First multiply £2 7s. 8½d. by 10 and then by 10 again.)

	£	s.	d.	
£2 7s. 8½d. × 10 =	23	16	10½	
£2 7s. 8½d. × 100 =	238	8	9	
£2 7s. 8½d. × 8 =	19	1	6	
£23 16s. 10½d. × 7 =	166	18	1½	
£238 8s. 9d. × 2 =	476	17	6	

Ans. £662 17 1½

Practice

In the past, many of the processes of simple arithmetic have been 'run to death' by applying them to examples which could be worked out much more easily in another way. It is far better that the child should show intelligence in choosing the tool which will allow him to perform the task presented to him in the neatest and most convenient way, than that he should be drilled in the use of particular tools applied to tasks for which they are not suited. The use of Practice is an instance of this. Many problems are worked by decimalization and multiplication more conveniently than by Practice, but a good pupil should have acquired such a facility with weights, measures and money and their various divisions that he can at once detect examples which are suitable for working by the latter process.

This method demands a fair knowledge of fractions and the aliquot parts of a pound (£) (or any other unit of weights or measures which is being considered).

The aliquot parts of a pound may be taught practically by using cardboard coins, though it is usually sufficient to obtain them by successive simple divisions.

10s. od. = $\pounds \frac{1}{2}$	4s. od. = $\pounds \frac{1}{5}$	6s. 8d. = $\pounds \frac{1}{3}$
5s. od. = $\frac{1}{4}$	2s. od. = $\frac{1}{10}$	3s. 4d. = $\frac{1}{6}$
2s. 6d. = $\frac{1}{3}$	1s. 4d. = $\frac{1}{15}$	1s. 8d. = $\frac{1}{12}$
1s. 3d. = $\frac{1}{18}$	1s. od. = $\frac{1}{20}$	10d. = $\frac{1}{24}$
$7\frac{1}{2}$ d. = $\frac{1}{8}$	8d. = $\frac{1}{10}$	5d. = $\frac{1}{48}$
	6d. = $\frac{1}{10}$	
	3d. = $\frac{1}{20}$	

Practice should also be given in recognizing other easy fractions of $\pounds 1$, e.g., 7s. 6d. = $\pounds \frac{3}{8}$, 12s. 6d. = $\pounds \frac{5}{8}$, 16s. 8d. = $\pounds \frac{5}{6}$, etc.

Example: $\pounds 2$ 7s. $8\frac{1}{4}$ d. $\times$ 278

		$\pounds$	s.	d.
$\pounds 2$	$\times$ 278	556	0	0
5s.	$\times$ 278 (5s. = $\pounds \frac{1}{4}$)	69	10	0
2s. 6d.	$\times$ 278 (2s. 6d. = $\frac{1}{2}$ of 5s.)	34	15	0
2d.	$\times$ 278 (2d. = $\frac{1}{16}$ of 2s. 6d.)	2	6	4
$\frac{1}{4}$ d.	$\times$ 278 ($\frac{1}{4}$ d. = $\frac{1}{8}$ of 2d.)		5	$9\frac{1}{2}$
$\pounds 2$ 7s. $8\frac{1}{4}$ d. $\times$ 278		662	17	$1\frac{1}{2}$

Care must be taken when working Practice sums *not* to add in the cost of the articles at $\pounds 1$ each, if this has been written down at the commencement of the working and is not required. It is recommended that an indication of the steps of the process should always be included, as this facilitates checking. (Practice methods may also be applied to problems on weights and measures.)

Division of Money

As in the cases of division mentioned previously, the twofold aspect of the process (partition and quotient) must be kept in mind.

Partition—when money is divided by number.

Quotition—when money is divided by money.

Long Division of Money

Divide £662 17s. $1\frac{1}{2}$ d. by 278.

(Divide pounds, convert remaining pounds to shillings, and add contents of shillings columns; divide the shillings, convert the remaining shillings to pence and add pence from pence column. Divide the pence and give remainder as pence, farthings, or fraction of a penny.)

	£	s.	d.
	2	7	$8\frac{1}{4}$
278)	662	17	$1\frac{1}{2}$
	556		
	106		
	20		
	2120		
	17		
278)	2137	(7s.	
	1946		
	191		
	12		
	2292		
	1		
278)	2293	(8d.	
	2224		
	69		
	4		
	276		
	2		
278)	278	($\frac{1}{4}$ d.	

Ans. £2 7s. $8\frac{1}{4}$ d.

In actual practice the above working may be considerably simplified. Top-form pupils who have gained facility with figures may employ the Italian method where only the remainders at each stage are written down.

Division by Factors

e.g., Divide £79 15s. 4d. by 63.

$$\begin{array}{r}
 \text{£} \quad \text{s.} \quad \text{d.} \\
 9 \overline{) 79 \quad 15 \quad 4} \\
 7 \overline{) 8 \quad 17 \quad 3} + 1 \text{d. remainder} \\
 1 \quad 5 \quad 3 + 6 \text{d. } (\times 9) \text{ remainder} \\
 \text{Total remainder } (1 + 54) \text{ d.}
 \end{array}$$

Ans. £1 5s. 3d. and 4s. 7d. remainder.

Quotition, or the Division of Money by Money

In this case each of the given sums of money is reduced to a convenient unit and division performed in the ordinary manner.

e.g., How many times may 7s. 3d. be paid from £5 8s. 6d.?
(3d. is a convenient unit.)

$$\begin{aligned}
 \text{£}5 \text{ 8s. 6d.} &= 434 \times 3\text{d.} \\
 7\text{s. 3d.} &= 29 \times 3\text{d.}
 \end{aligned}$$

$$\text{Required number} = \frac{434}{29} = 14 \text{ and } 28 (\times 3\text{d.}) \text{ remainder}$$

Ans. 14, and 7s. 0d. remainder.

Decimalization of Money

It is often convenient to express sums of money in pounds and decimal fractions of a pound.

$$\begin{aligned}
 2\text{s.} &= \text{£} \frac{1}{5} = \text{£} \cdot 1 \\
 1\text{s.} &= \text{£} \frac{1}{20} = \text{£} \cdot 05 \\
 6\text{d.} &= \text{£} \frac{1}{40} = \text{£} \cdot 025 \\
 (3\text{d.}) &= \text{£} \frac{1}{80} = \text{£} \cdot 0125 \\
 24 \text{ Farthings} &= 6\text{d.} = \text{£} \cdot 025.
 \end{aligned}$$

Thus the error in reckoning $\frac{1}{4}$ d. as £.001 is 1 in 25, or if we wish to present the decimal *correct to 3 places* we may

count to 12 farthings, reckoning $\cdot 001$ for each farthing, but over twelve farthings an extra $\cdot 001$ must be added.

$$\text{i.e., } 2\frac{3}{4}\text{d.} = \text{£}\cdot 011 \text{ (correct to 3 places)}$$

$$3\frac{3}{4}\text{d.} = \text{£}\cdot 016 \text{ (correct to 3 places)}$$

If a sum of money is expressed in decimal form correct to 3 places, the maximum error is half a farthing above or below the given amount, i.e., if we are given $\text{£}2\cdot 518$ correct to 3 places, the actual value must lie between $\text{£}2\cdot 5175$ and $\text{£}2\cdot 5185$. A true idea of this possible error must be grasped, for it will become greater or less according to the processes of multiplication or division in which the decimal is subsequently involved.

It is not necessary to say anything here about English weights and measures. Senior School work will consist of an extension of that of the Primary School outlined in Chapter III. Many of the processes which we have given for use in money problems are also applicable to those concerning weights and measures. The tables should be reduced to a minimum, elaborate processes of reduction should be avoided and a practical bias maintained.

Time and its Measurement

Ancient peoples attempted to measure days, months, seasons and years long before they understood the simple astronomical phenomena which determine these periods of time, and also before any suitable mathematical notation had been discovered for recording them. The idea of time 'like an ever rolling stream' is accepted without thought to such an extent that many adults express surprise when they come to read books about the matter, and find that there are enormous difficulties in the way of measuring time, and that it is a difficult or impossible task to disentangle time and space.

The simple facts about the earth revolving on its axis once

a day, the moon going round the earth once in a lunar month, and the earth going round the sun once in a year are easily demonstrated with simple models. In practice, matters are much more complex than this for all these motions are subject to variation and the orbits of neither earth nor moon are perfect circles, and thus astronomers have to take many observations and do much calculation to find the average day and month. From the historical point of view the difficulties arose because the lunar month is not an exact multiple of the day and the year is neither an exact multiple of the month nor of the day. The Egyptians tried to overcome the difficulty by adding five days to the lunar year, and their only check on the accuracy of their estimates was by observing the return of the seasons.

The Greeks (about 600 B.C.) suggested that the months should be alternately 29 and 30 days. When astronomy could use a better system of recording, observations on the stars were made and it was then imagined that the year was 365 days.

About a century before the birth of Christ, Hipparchus was able to find the difference between the Solar day and the Sidereal day, which is about four minutes. The Solar day is the time taken for the earth to revolve once on its axis as measured by the sun; and as a simple demonstration, with a ball of wool through which is thrust a knitting needle (revolving on its axis and also going round an orbit) will show, such a day is slightly different from one measured by observations on 'fixed stars' outside the solar system.

It must be remembered that the ancients did not all use the heliocentric theory (i.e., that the sun is the centre of the solar system) which developed later from the work of Copernicus and Galileo, and although it is believed that some of the great ancient philosophers may have anticipated it, their achievements in astronomy appear all the more remarkable. The various references in the Bible to the

dividing of time are interesting; we pass from the idea of a day from sunrise to sunset, to a *division of the night into watches*.

The Romans divided the day into 12 parts and later the process was extended so that the whole day from midnight to the following midnight was divided into 24 parts (an idea first expressed by Hipparchus). The sundials and water clocks of surprising ingenuity of construction which were then in use, were incapable of measuring much more accurately than halves and large fractions of an hour. The Babylonians used the sexagesimal system of notation and it is probable that this was the origin of the practice of dividing the hour into 60 minutes (minute—a very small part). Later, when astronomical observations attained greater refinement, a still smaller division was employed, which was known as the second minute, and afterwards the second.

The names of the days find their origins in the names of various deities, though some races merely gave them numbers. The Romans used the names of members of the solar system:

Solis, Lunae, Martis, Mercurie, Jovis, Veneris, Saturni. The similarity to the French: Lundi, Mardi, Mercredi, Jeudi, Vendredi, Samedi, is obvious.

The chief difficulty which arose in the making of calendars was due to the fact that there are not 365 days in a year but nearly $365\frac{1}{4}$. The addition of a day every fourth year (leap year) helps to solve the difficulty, but a system which allows only 24 leap years to the century, or even the more accurate number of 97 leap years in four centuries, leaves an error of only about half an hour each 100 years. The word calendar is derived from the Roman *calends* meaning the beginning of the month; almanack is from the Arabian *Al manac*—the diary or daily account.

The Roman Calendar originally consisted of the months beginning with March. In the eighth century B.C., January

and February were added and an extra month was inserted once every three years. The seasons were badly misplaced owing to this, and accordingly Julius Cæsar used a leap year once every four years, and renamed one of the months July, to commemorate his action. (His successor, Augustus, also left his mark on the calendar, by renaming the following month.)

This Julian calendar was used until the sixteenth century, when Pope Gregory XIII made some modifications.

The Gregorian Calendar. Instead of counting from the foundation of Rome, the beginning of the era dated from the birth of Christ. (Since Gregory's time further adjustments have been made; and it must be remembered that the exact number of years since Christ's birth is not given by the number of the calendar year.) Julius Cæsar's correction for leap years still left an error which had accumulated during the centuries to become 10 days by the end of the sixteenth century. The correction was not made in England until 1752, when the day that followed the 2nd of September was called the 14th. Ignorant people, thinking that eleven days' life and wages had been taken from them, rioted.

The entire story of the measurement of time in various countries and civilizations is most interesting. Throughout the history of astronomy the desire to measure time more accurately, particularly for maritime and scientific purposes, has given the greatest encouragement to the inventors of clocks, optical instruments and other apparatus used in observations.

A graduated course of school work on time:

1. *Infant School.*

How to tell the time. How to talk about the time. Practice with clock dials. Days and months. Rhymes about the seasons, days and months. Simple counting of hours during the day, of weeks and months from the calendar.

Simple exercises in estimating time, e.g., how long does it take me to dress, to eat my tea, to come to school? Children are always interested in experiments with calendars, which they are able to make in 'craft' work.

2. *Primary School.*

A revision and extension of work taken at the Infant School stage. A.M. and P.M. and what they mean. Practice in estimating times. Dividing up the days of our life. How the days and months got their names. Further work with the calendar to estimate times between events in days. Time-tables and what they tell us. Finding the best train or 'bus and 'hunting' for suitable connections. Early ideas of speed. Roman figures. Various kinds of clocks. The 24-hour day in Europe. Estimating how far I can walk and what I can do in other ways in a given time. Estimating the time which is likely to be taken in performing a simple task.

3. *Senior School.*

An extension of the above. Things that may vary with time. Graphs to show velocity and acceleration. Power. Various types of clocks and how they work. The earth as a clock. Greenwich mean time. Wireless time signals. The seconds pendulum. (Time of swing is dependent only on length and gravity.) The work of Galileo and Huygens. Importance of time to navigators. How to calculate longitude. (Problems such as: 'Find at what time the hands of a clock will be at right-angles between 2 and 3 o'clock have no practical use and may be excluded.') The mathematics teacher should obtain the help of his science colleague who will be able to supply many examples from dynamics, heat, light, sound and electricity. The teacher should read books such as *The Nature of the Physical World* (Eddington) and *The Mysterious Universe* (Jeans) where modern ideas about time and space are simply expounded.

The Metric System

The value of a system of weights and measures, which could be referred to a single standard, and capable of being manipulated simply by the use of decimals, is easily seen. Long 'conversions' and 'reductions', as in English measures, are avoided for a simple movement of the decimal point will suffice; division and multiplication can be conducted by the use of the standard rules, using decimal fractions; and rough calculations and estimations of errors are easily negotiated. Again, there is no need to learn cumbrous tables of weights and measures, which have been derived from a large number of unrelated sources.

The Metric system, as we know it, dates from the French Revolution, and may be regarded as a means of establishing uniformity in weights and measures, and as a method which avoided secrets and obscurity, and could be used by everybody who was able to perform simple operations with decimals. The scheme was formulated by Lagrange (an Italian-born mathematician, who lived and taught in Paris) and it was adopted in 1799.

The impetus given to the sciences by the Metric System, and by the labours of the Commission who were responsible for its inception, was felt for half a century or more in the physical sciences in France. The Metric System has undeniable advantages and for scientific measurements its use is practically universal. If our scale of notation were 12 instead of 10, its value would have been even greater, for the number 10 is not ideal for use in expressing simple fractions. Herein lie any disadvantages which the Metric System may possess; for we still adhere to our thirds, and quarters; and our small fractions in the money, length and area tables perform exactly what is required of them. Through long custom in ordinary buying and selling we tend to think of fractions, not as decimals but as $\frac{1}{2}$, $\frac{1}{4}$, $\frac{3}{4}$ and

$\frac{1}{8}$, etc. Now 10 does not divide by 4 easily, neither does 100 by 8.

The unit from which all the other measurements are derived is the Metre. This was originally intended to be a *ten-millionth part of the earth quadrant (i.e., the distance from a pole to the Equator)*, but more recent investigations have shown that ideas about the shape and size of the earth, which were held at the end of the eighteenth century, were not quite correct. Nevertheless, the Metre is now defined as the distance between two fine marks on a carefully-preserved standard bar kept at a certain temperature. As not even metals are absolutely permanent in shape, it is interesting to note that during the present century a most accurate determination of the length of the Metre has been made in terms of the wavelength of the red light emitted when cadmium is heated in the electric arc.

From the Metre, which is approximately 39.37 inches, we get the tenth of a Metre or decimetre, which is nearly four inches, the centimetre ($\frac{1}{2}$ of an inch approximately) and the millimetre ($\frac{1}{25}$ of an inch approximately).

(It is interesting to note that a pendulum of length one Metre swings from one side to the other in approximately one second.)

The multiples of a Metre:

The Dekametre (nearly 11 yards).

The Hectometre (nearly 110 yards or half a furlong).

The Kilometre (nearly $\frac{1}{2}$ of a mile).

For our purpose the Kilometre is the most important of these multiples; and little is lost by omitting throughout work on the Metric System, special terms for the 10 and 100 multiples.

The derivation of the Square measures is simple. An Are (area) is a square of side 10 metres and its area is therefore 100 square metres or nearly 120 square yards. It is rather less than a fortieth of an acre, or four square poles. (Notice

that as we are now squaring the former units with the 10's relationship of the linear measure, here the ratio is 100.)

Cubic Measure

1. Cubic millimetre (used by scientists). As a thousand of these make 1 cu. centimetre, this is very small. A drop of water may contain from 10 up to 40 cu. millimetres or more.

2. Cubic centimetre. This is very useful for scientific purposes and is abbreviated to c.c. As $\frac{1}{6}$ cms. = 1 inch approximately, $(\frac{1}{6})^3$ c.c.s. or $15\frac{1}{2}$ c.c.s. (approx.) are contained in 1 cu. inch.

3. Cubic decimetre (1000 c.c.s.) or a litre. This is a very convenient measure of capacity. It is almost equal to $1\frac{3}{4}$ pints. 5 litres would be $\frac{3}{4}$ pint more than a gallon, and would be a useful measurement for petrol and oil. $\frac{1}{2}$ litre is rather less than a pint, and $\frac{1}{4}$ litre is a very popular measure for light wine and other beverages sold abroad.

4. Cubic metre (about $1\frac{1}{3}$ cubic yards or 36 cubic feet). This contains 1000 litres. An ordinary cartload of soil would be of the order of a cubic metre. In actual practice people who have to measure liquids for commercial purposes would usually prefer to use litres.

Units of Weight. There is no need at this stage to take up time in showing the connection between mass and weight. It will suffice to define a gram weight as the *weight* of a cubic centimetre of pure water, when it is most dense. (It will probably be possible to make reference to a little mechanics and physics in this connection.)

The value of this connecting link between length, area, volume and weight (more strictly mass) will be seen when problems on density and specific gravity are worked later. A kilogram (1000 grams, or the weight of a litre of water) is approximately 2.2 lbs., therefore 28 grams are almost equal to an ounce. A ton (2240 lbs.) is rather more than 1000 kilograms. An 8-stone boy weighs about 50 kilograms.

Money. Many European countries and America use a decimal system for money. Of course, there cannot be a fixed monetary system related to the other standards, such as the price of a gram of gold, as the value of money varies with the prices of commodities. The systems which will be discussed are American—dollar and cent.; French—franc and centime; German—Reichmark and pfennig.

Before working any problems on the conversion of money from one system to another it will be useful to discuss the approximate current prices of common commodities in each country, e.g., a newspaper, an orange, a lunch, a textbook, a ball, a bicycle, a gallon of petrol, a railway journey of 50 miles or its equivalent in kilometres. Practical examples of the conversion of francs into pounds, pounds into dollars, etc., may then be worked. The teacher should deal briefly with various currency problems, the gold standard, how and where money is converted, how commission is charged, German currency restrictions, special cheap marks for travellers and how the system works, the history of currencies, etc. Newspapers give daily accounts of the value of the pound in various countries.

Many other simple and quick conversions for various measures may be evolved, e.g., $100 \text{ millimetres} = 4 \text{ inches}$ (approx.). Therefore to convert millimetres to inches, multiply by 4 and move the decimal point two places to the left.

CHAPTER VIII

ALGEBRA, GRAPH-DRAWING AND LOGARITHMS

Algebra

IN the past, algebra has tended to be an abstract subject dealing chiefly with the last three letters of the alphabet by mysterious processes. If the subject is made to grow out of arithmetic, if it is seen that it contains extensions of old knowledge which are of the utmost importance from the utilitarian point of view, and finally if the mystery is removed from its name and its symbols by reading a little of its history, no serious troubles are likely to arise. Actually, in his manipulation of simple formulæ used in mensuration, the pupil will have been using algebraic processes without realizing it.

The word algebra is a corruption of the title of the Arabic treatise on the subject, 'al-jabr m'wa'l muquabalah', which relates to two operations in solving a simple equation. (Al in Arabic is equivalent to 'The' in English; 'jabr' refers to the important operation of taking a quantity from one side of an equation to the other and changing its sign, whilst 'muquabalah' denotes the process of subtracting similar quantities from both sides of an equation.) The pupil will already be familiar with the use of symbols in arithmetic and simple mensuration, e.g., H for height, L for length, etc., and he will readily appreciate that algebra, in expressing general truths in addition to the more particular ones of arithmetic, must concern itself with symbols.

We have already referred to milestones in the history of mathematics which are of the utmost importance. One of these was the expression of a clear idea of algebraic symbolism in the *Logistica Speciosa* by a Frenchman, Franciscus

Vieta (*c.* 1590). In this work he says, 'we are aided by an artifice which permits us to distinguish given magnitudes from those which are unknown or sought, and this by means of a symbolism which is permanent in nature and clear to understand, for instance, by denoting the unknown magnitudes by A or any other vowels, while the given magnitudes are designated by B, C, G, or other consonants.' Actually, Vieta's System only lasted for about fifty years, for Descartes used the first letters of the alphabet for the given (known) quantities and the last letters for the unknown. This system has survived to the present day—hence our use of x , y and z .

Algebra should commence by consolidating and extending the simple symbolism and linear equations of arithmetic. It is fatal to interest to begin with a systematic treatment of the first four rules in terms of x , y , z , and the products and powers thereof; and indeed, even in Secondary Schools much of the abstract algebra (such as complex algebraic fractions, which can only be justified by saying that they give practice in algebraic manipulations), has given place to more useful aspects of the subject. There is a general feeling that the calculus should be started earlier. This subject, the importance of which it is difficult to exaggerate, is outside the scope of Central School work; but it is certainly of the greatest use to students who are passing on to evening institutes for courses in engineering and the applied sciences. In Secondary Schools it could be commenced earlier, both for its value in physics, mechanics and chemistry, and in paving the way for later pure mathematics.

Under these circumstances it seems a pity that graph-drawing tends to be discontinued just at a point where it could be made a most vital study, and that the idea of a function of x —a cardinal step in useful mathematics, and one which is taught at a quite early stage in German schools—is overlooked. In Senior Schools, algebra might be regarded as an extension of arithmetic and as a connecting link in

graph-drawing, geometry, trigonometry, mechanics and simple science, and possibly as a preamble to the calculus in later technical schools. In the past, there has been a tendency to allow the essential nature of algebra to become obscured by masses of mechanical examples of the standard processes of algebraic manipulation. It must be stated, however, that failure to perform these competently will retard later work, and that in some Secondary Schools the reaction has proceeded too far. Factorizing, the rules analogous to those of arithmetic, the simplification of algebraic fractions, and all the devices of elementary algebra are only a means to an end, and that end is the manipulation of algebraic formulæ and equations to show new relations and further truths. It is impossible to deal with graphs and variation, with the changing of the subject in a formula, with the problems arising in logarithms, with simple algebraic methods applied to arithmetic, mensuration, geometry and trigonometry, etc., without being familiar with the common algebraic processes—hence their importance.

The first important point is a discussion of the use of symbols and signs as a shorthand, and the use of formulæ as a way of expressing concisely and without ambiguity what can only be said at length, and probably with some clumsiness of expression, in words. The story of algebra will help to make this clear and children will see why the Greeks, despite their superb achievements in deductive geometry, were poor algebraists.

Algebraical language grew up slowly¹ :—

$$4 \text{ Census et } 7 \text{ demptis } 13 \text{ rebus aequatur zero} \quad (1464)$$

$$4 \text{ Census p } 7 \text{ de } 13 \text{ rebus ae } 0 \quad (1494)$$

$$4 \text{ in A quad} - 13 \text{ in A plano} + 7 \text{ aequatur } 0 \quad (\text{Vieta } 1591)$$

$$4^2 - 13^1 + 7 = 0 \quad (\text{Stevinus } 1585)$$

$$4x^2 - 13x + 7 = 0 \quad (\text{since the time of Descartes, } 1637)$$

¹Adapted from Sullivan's *History of Mathematics in Europe*.

To a considerable extent algebra may be regarded as a branch of logic; and although discussions of the nature of mathematical thought are outside the scope of Central School work, it is essential that the teacher should think about these problems, for they are bound to affect his attitude to the subject. (From an elementary standpoint nothing could be better than the introductory chapters of *The Teaching of Algebra* by Sir Percy Nunn. This might be followed by M. Black's *The Nature of Mathematics* in which is presented in intelligible form an account of the work of various mathematical philosophers, including Lord Russell's *Principles of Mathematics*, the original form of which gives difficulty to many mathematicians.)

A start may be made with simple formulæ and very easy linear equations. Take, for instance, $A = L \times B$, for the area of a rectangle. This may be regarded as a generalization summing up the areas of all rectangles. However, it is more than a generalization. Its truth does not rely on all the cases of areas of rectangles which in the past have been examined, for it is possible to 'bring to light the essential process concealed in a particular or accidental numerical garb',¹ which is a process of analysis. The distinction between 'generalization' and 'analysis' should be kept in mind. Arithmetic, mensuration and simple science will supply many cases of symbolism and formulæ: e.g.,

(a) The areas of squares, rectangles, triangles, circles, and the volumes of rectangular solids, spheres and cones, etc.

(b) Pythagoras' Theorem and simple trigonometry; formulæ for money; compound interest; profit and loss.

(c) Ohm's Law; capacity for heat; heating effect of a current; power; work, specific gravity; proportional division; simple and compound Proportion; Variation.

(d) General expressions for numbers and number series.

¹Nunn, *The Teaching of Algebra*.

A few simple formulæ may be constructed and the various conventions as regards sign and symbol discussed. Algebra will grow up with trigonometry and graph-drawing, each helping the other.

Simple transformations of practical formula will serve to show their application, e.g., Ohm's Law $C = \frac{E}{R}$ when the 'subject is changed' becomes $R = \frac{C}{E}$ or $E = CR$.

(C = Current, R = Resistance, E = Voltage drop along a piece of wire.)

The combining of the formula is also of the greatest importance.

Energy lost through heating effect of a current passing through a wire $= C \times E$ (per second)
 $= C^2 R$.

This expression containing the 'Square of the current' is of wide application in electricity, and at once shows why in the transmission of electricity a high voltage and a correspondingly small current are desirable. The steps should be noticed. By analysis or generalization a formula is obtained which states in precise language certain truths about external things. The formula may be manipulated or combined with others, and new relations appear. We then expect to find some objective and practical counterpart to the results of our manipulations of the symbols.

Equations

Equations to find unknowns are a valuable part of algebra, but in the past the formal solution of equations has occupied far too large a part of the work. From one point of view some, such as simultaneous equations with two or more

unknowns, have few applications. It is recommended that only simple equations should be used; but there should be an endeavour to investigate intelligently every step towards the solution.

Simple linear equations will first be considered, and many examples of these may be found from practical arithmetic. Selections of simple historical examples from Hindoo and other sources are always interesting. If the meanings of the $=$ sign have not been thoroughly examined before, they should be considered now. This sign may mean 'identically equal' or 'equivalent'. In algebra it has several implications. In equations it means that the sides are equal for a single or limited number of values of the unknown. Compare the uses of the $=$ signs in the following examples:—

$$\begin{array}{ll} (a + b)^2 = a^2 + 2ab + b^2 & \text{(identity)} \\ x^2 + 4x + 4 = 0 & \text{(equation)} \\ \text{The height of the pole} = 20 \text{ ft. (is).} & \end{array}$$

The 'balance method' of dealing with equations commends itself to children, but it often leads to unintelligent assumptions. 'Treat both sides in the same way' is a rule which is apt to be dangerous: for instance, dividing both sides of the equation by the unknown or the function of the unknown will remove what may be the most important root. Squaring and 'square-rooting' both sides may catch the unwary by adding or subtracting roots. Nevertheless, the balance method will serve to show how the unknown quantity may be isolated on one side, and the known numbers on the other.

Take the case of a butcher weighing a piece of meat. He puts the meat on one pan and a 4-lb. weight on the other. The 4-lb. weight is slightly too heavy, so to save himself the trouble of building up the correct weight on that side he puts a 2-oz. weight on the pan which holds the meat.

$$\text{Weight of meat} + 2 \text{ oz.} = 4 \text{ lbs.}$$

Subtracting 2 ozs. from both sides, weight of meat = 4 lbs.
 - 2 ozs. = 3 lbs. 14 ozs.

The see-saw (with equal arms) also appeals to children, but it is sometimes difficult to avoid the question of moments.

Weight of Kathleen (6 stones) + weight of John ($3\frac{1}{2}$ stones) = weight of Mary (5 stones) + weight of Ruth.

Find the weight of Ruth.

It will follow that the rule of 'take the term to the other side and change its sign' is merely the equivalent of adding or subtracting similar quantities to or from both sides.

More advanced equations will be taken when there is some facility with simple algebraic manipulations. From the utilitarian point of view simultaneous equations have lost some of their importance. All equations must also be considered from a graphical standpoint. If the early work on simple equations, formulæ, and symbolism has been understood it will go far to prevent a common fault in the unintelligent working of problems by the use of equations. This error arises from the equating of terms and numbers representing quantities of different denominations.

The fundamental laws of algebra may be illustrated first by considering simple numbers arising in arithmetical operations. By generalizing, extensions are made to include symbols and the laws may be demonstrated by reference to the manipulation of formulæ obtained from mensuration. There is no need for the child to know the names of the following processes.

Law of Association:

$$\begin{aligned}(a + b) + c &= a + (b + c) \\ (a \times b) \times c &= a \times (b \times c) \text{ etc.}\end{aligned}$$

Law of Commutation:

$$\begin{aligned}a + b + c &= a + c + b = b + c + a \\ a \times b &= b \times a\end{aligned}$$

Law of Distribution:

$$a \times (b + c) = a \times b + a \times c$$

$$(a + b)(c + d) = a \times c + a \times d + b \times c + b \times d$$

The usual algebraic conventions will be noticed and practice is necessary to obtain facility in reading the new language and manipulating the symbols. Reference by analogy to pure arithmetic and simple mensuration will often help to make a point clear.

Directed numbers and algebraic addition are not difficult to understand if a few practical cases are used.

A line may be drawn on the floor and a point selected in the middle of it as a point of reference. Distances in any direction may be called positive and in the opposite direction negative; or the line may be supposed to be drawn through various towns on a map, and 'positive' and 'negative' distances travelled along the line. If I set out to go to a town 20 miles away which is due north, but instead mistake the direction and go due south for 5 miles, as regards attaining my objective, this will be a journey of -5 miles. If I now realize my error, turn round and walk for 12 miles in the correct direction, I have gone $(-5 + 12)$ or 7 miles towards the town from my original starting point. Many simple applications of directed numbers will serve to show how signs are given to numbers in algebra:—Dates B.C. and A.D.; temperatures below and above zero; debts and credits; adverse trade balances; distances below and above water-level; the weight of a balloon; a cork rising to the surface of the water; a mine-shaft sunk from an elevated plateau, etc.

Simple addition and subtraction of directed numbers may be demonstrated by drawing lines; and throughout the early work in algebra, the geometrical counterparts should be noted. The multiplication and division of directed numbers is a matter of slightly greater difficulty.

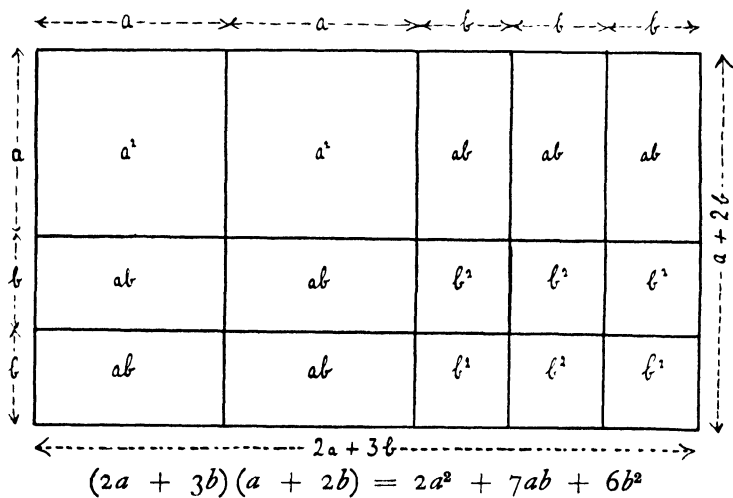
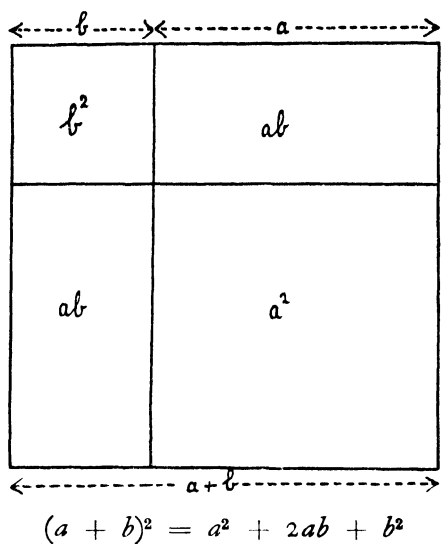
$$\begin{aligned}
 a \times b &= ab \\
 -a \times b &= -ab \\
 a \times -b &= -ab \\
 -a \times -b &= ab
 \end{aligned}$$

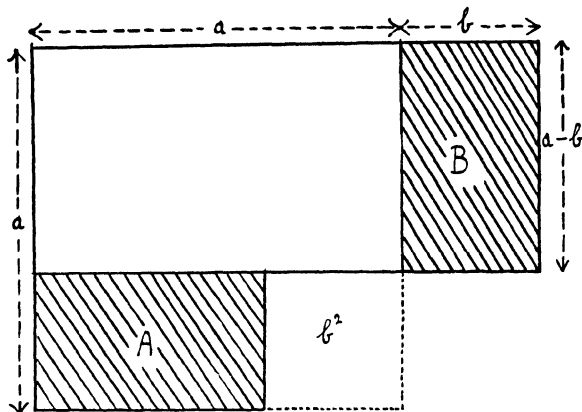
The phrase 'two minuses makes a plus' is misleading and should not be used. We do not prove that $-a \times -b = ab$ but we can give demonstrations to show that its use gives consistent results. The laws of division of directed numbers may be found by extending those for multiplication. The multiplication of negative directed quantities is often demonstrated by considering a train travelling due north (which for the purpose of the argument may be taken as a positive direction). Suppose it goes through a midland city N at 12 noon (zero hour) and that its speed is a miles an hour. In b hours after noon it will be ab miles north of N. In b hours before noon (i.e., $-b$ hrs.) it was ab miles south of N, i.e., $-ab$ miles north of N.

Now consider the train running at the same speed in the opposite direction and going through N at 12 noon. If going towards the north is the positive direction, its speed is now $-a$ miles an hour. In b hours after noon it will be $-a \times b$ miles north of N (ab miles south of N). In b hours before noon ($-b$ hours) it will be $-a \times -b$ miles north of N or ab miles north of N. A graph of the two journeys will make the matter easy.

Identities, factorization, squares, and square roots may be demonstrated by drawing rectangles and squares, and if necessary, continuing this by paper-cutting.

If the shaded portion A is cut from the bottom of the figure representing $a^2 - b^2$, and placed in the position of B, a rectangle of sides $(a + b)$ and $(a - b)$ is formed (page 117).





$$a^2 - b^2 = (a - b)(a + b)$$

Many more instances may be given to children. They will enjoy experimenting with large and small squares and rectangles. If crayons or coloured papers are used for identifying each of the sections, more difficult factorizations and identities may be attempted. It is quite possible, using blocks of plasticine cut with a knife, or better, wooden blocks divided up by sawcuts, to demonstrate such identities as:

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

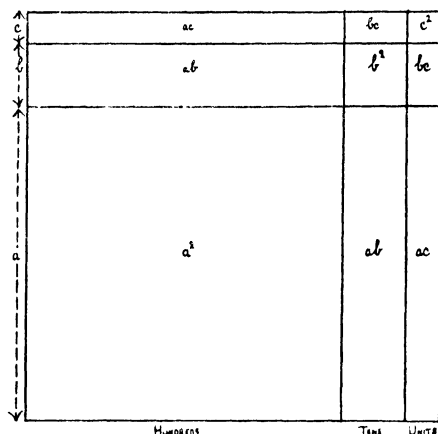
$$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$$

Square Roots and Squaring

Square roots will have been introduced by considering the areas and sides of squares. Square numbers will have been factorized and the factors paired.

The 'long division' method of finding the square root may be illustrated by a diagram.



$$(a + b)^2 = a^2 + 2ab + b^2$$

or to take particular instances

$$(31)^2 = (30 + 1)^2 = 900 + 60 + 1$$

$$(37)^2 = (30 + 7)^2 = 900 + 2 \times 7 \times 30 + 49$$

or to take an example with three 'digits'

$$(a + b + c)^2 = a^2 + b^2 + c^2 + 2ab + 2bc + 2ca.$$

The successive approximations to the square root with each stage of the working, and the reason for 'doubling' the figure in the answer at each division are readily seen from the diagram.

The use of the radical sign $\sqrt{\quad}$ should be encouraged and the following expressions understood:

$$\begin{aligned} \sqrt{a} \times \sqrt{b} &= \sqrt{ab} \\ \sqrt{a} \times \sqrt{a} &= a \end{aligned}$$

Further elucidation of these matters will come when index notation and indices are used. Indices may be approached through the mensuration of squares, cubes, and other figures and solids, and by simple experiments with powers of 2.

$$a^1 = a, \quad a^2 = a \times a, \quad a^3 = a \times a \times a$$

$$10^1 = 10, \quad 10^2 = 100, \quad 10^3 = 1,000, \quad \text{a million} = 10^6, \text{ etc.}$$

$$a^3 \times a^5 = \underbrace{a \times a \times a}_{a^3} \times \underbrace{a \times a \times a \times a \times a}_{a^5} = a^8$$

In the general case $a^n \times a^m = a^{n+m}$.

$$(a^3)^4 = (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a) \times (a \times a \times a)$$

$$\times (a \times a \times a) = a^{12}$$

In the general case $(a^n)^m = a^{n \cdot m}$

$$a^5 \div a^3 = \frac{a \times a \times a \times a \times a}{a \times a \times a} = a^2$$

In the general case

$$a^m \div a^n = a^{m-n}$$

$$\sqrt[3]{a^6} = \sqrt[3]{a^2 \times a^2 \times a^2} = a^2$$

In the general case

$$\sqrt[n]{a^m} = a^{\frac{m}{n}}$$

$$N.B. \quad \sqrt[2]{a^4} = a^2$$

If these simple principles are extended to fractional and negative indices it is found that they still yield consistent

results. Indices are useful when we deal with logarithms. Index notation is valuable for expressing very large quantities, such as the distances of the fixed stars, the speed of light, etc.; or very small quantities such as the 'wavelength' of X-rays, the size of atoms or bacteria, etc. Archimedes was feeling for a suitable index-notation when he showed how it might be possible to estimate the number of grains of sand necessary to make a 'world of sand'. (The sand-reckoner.)

Index notation is so simple that it is easy to write down symbols which represent numbers far beyond human comprehension. Numbers written in index notation are compared by multiplication and not by addition and subtraction. This holds many pitfalls for the uninitiated, e.g., the difference between 10^6 (a million) and 10^4 is $\frac{99}{100}$ of 10^6 , or only $\frac{1}{100}$ of the whole less than the million. With simple figures such as the above the principle is easily seen, but many come to grief when dealing with astronomical distances on the one hand, and microscopic measurements on the other.

In this section there has been no attempt to deal systematically with the topics of elementary algebra. The work will proceed side by side with more advanced arithmetic which will benefit by algebraic treatment, and with mensuration, graph-drawing and trigonometry for which it is absolutely essential.

Graph-drawing

All children will have seen a graph of one kind or another, such as a temperature chart, a newspaper advertisement showing rise in sales, an automatically recorded barograph, etc. The idea of variation is a fundamental one; and involving, as it often does, difficult conceptions of cause and effect and correlation, it is not easy to grasp its full importance. The value of graphical work does not require stressing here, for since Professor Perry championed, at the end of last

century, the value of such modes of expression in school mathematics, it has steadily gained favour in school work.

Descartes is usually credited with the invention of the graph, but in many ways the Greeks anticipated him, and if they had possessed a better idea of algebra it is probable that they would have perfected the method of graphs.

Graphs are useful in schools because:

1. They make an immediate appeal to the eye. Even when the student has little aptitude for dealing with figures, the visual image is the one above all others, which can most usually be remembered, analysed, and interpreted.

2. Graphs give a picture of the variation of one quantity with another. 'The invention of variables', says Sir Percy Nunn, 'was perhaps the most important event in human evolution. The command of their use remains the most significant achievement in the history of the individual human being.' Thus, graphs are the means of setting down in a simple way the manner in which quantities vary with one another. Time-graphs can be found for many things which happen in our lives, in mechanics, engineering and practically all the sciences. The engineer draws a velocity-time curve, the electrician a voltage or power-time curve, and the biologist a growth curve. The graph often enables investigators to discover the law of the variation of one thing with another or, indeed, to discover whether there is any variation or correlation at all. It allows interpolation (the finding of intermediate values), and extrapolation (the extension of the curve by having regard to its general shape, and hence finding further values which are outside the range of those which are plotted). The two latter processes are not to be undertaken lightly, for unless the mathematician is certain of the shape of the graph it is unwise to expect to obtain intermediate or extra values.

3. The graph may be used as a ready-reckoner, a timetable, or a means of making quick conversions from one

system of measurement to another and from one scale to another.

4. Even early work in mathematics, mechanics, or the sciences constantly supplies instances of changes which take place systematically during periods of time. Until something is known of the calculus, of exponentials, etc., the graph is often the only means available for representing the nature of the variation. (The teacher will realize that although the representation of numbers by points on lines appears to be obvious, certain assumptions have had to be made. Intuitively the matter appears to be one of common sense, but it must be stated that much of the labour of nineteenth-century mathematicians was devoted to rigorous examinations and proof of many of the assumptions which former mathematicians had not been able to investigate.) Thus the graph represents an excellent way of approaching later work in the calculus. Finding the areas enclosed by graphs is an easy way of 'integrating'; tangents drawn to points on graphs anticipate differentiation; and maximum and minimum points on curves are readily seen and interpreted. The idea of a function of x , which might be introduced as early as possible, can be demonstrated by graphical means.

In Senior School work, a start may be made with column graphs, which by vertical lines, drawn to suitable scales, represent various statistics, such as the numbers in different classes of the school, imports, exports, income-tax during various years, rainfall during various months, prices of commodities, heights of plants on various days, etc. In some schools the barometer is read every morning and the barometric pressure is noted by drawing equidistant vertical lines. If the tops of these lines are joined by straight lines we get a graph, but it is obvious that this line cannot be expected to yield a continuous picture of the changes in barometric height during a certain period. If we read the barometer at

shorter intervals of time, say every hour, our barograph will be made much more complete; but if we can obtain a recording-barometer which plots its own graph from one instant to the next, we obtain a true pictorial record of all the barometer readings during the period. Again, a boy may plot his height every month and make a graph. Here he may read intermediate values (interpolate) with some degree of certainty because he knows that he grows taller every month and that his increase during such a period is not great.

The dangers of unintelligent interpolation may be seen by plotting a graph to show the altitude of the sun at midday every day throughout the year. The graph made by joining up successive points representing the daily angle of the altitude of the sun when it crosses the meridian is a simple curve, but a moment's thought will show that to take intermediate values on the graph between one midday and the next (the altitude at 10 p.m. for instance) will yield the most ridiculous results.

Several graphs of simple statistics which may be collected from newspapers, from *Whitaker's Almanack*, from the Registrar-General's reports, or from the child's own observations, should be drawn and interpreted; sudden rises and falls in the graph should be noted and the reasons sought. As an alternative to this, graphs representing certain phenomena may be given to children and they should be asked to suggest explanations for the shapes of the graphs. Graphs showing the supply of electric current given by a power station, road accidents, population in certain districts, health of children, and their attendance at school, the amount of milk given by a cow, the speed of a bicycle or car during a journey, etc., may be examined and the characteristics discussed with reference to their causes. Sometimes the graphs, when intelligently considered, suggest means for the improvement of performance in future. It is impossible

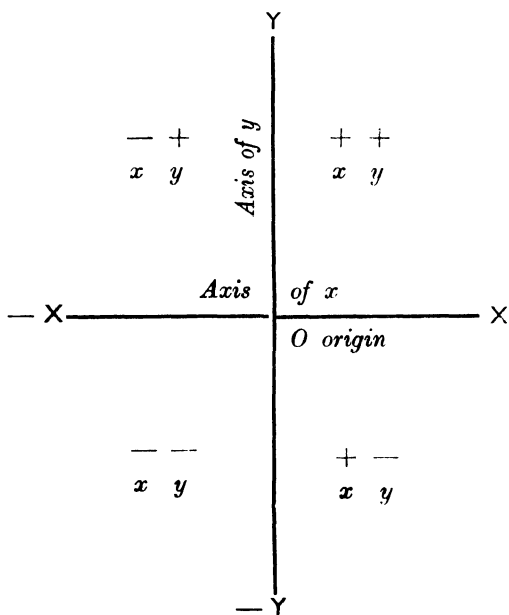
to exhaust the number of uses of graphs of this type which may be drawn, for any quantity which varies with another may be recorded in graphical form. The teacher wishing to know whether his children are graded correctly or whether his examination questions are satisfactory and properly marked, may draw a graph of his results; a scientist wishing to calibrate an instrument draws a graph of its possible error at every point; a doctor may need a graph for the purpose of diagnosis and so on.

'Mathematical' Graphs

The term 'mathematical' graphs is usually applied to those for which there is a simple algebraic formula. Often 'statistical' graphs may have a formula found for them and, indeed, one of the most valuable means by which researchers in the applied sciences try to find formulæ is by plotting graphs (sometimes taking the logarithms of one of the variables before plotting, because when only a limited range of numbers is considered many things in the sciences may be considered to vary logarithmically).

From the early graphical work the pupil will have seen how to plot a point by using its distances from two arbitrary axes, but his work will have been confined to positive numbers. This may be made more interesting by giving distances from walls, paths, trees, etc., as clues for buried treasure. Hence the child will be able to find the minimum amount of data necessary to determine the exact position of a point (e.g., distances at right-angles from two fixed lines; distance in a certain direction from a given point). By drawing two axes so that the paper is divided into four approximately equal rectangles we can extend our graphs so that they may be used for negative numbers. The point of intersection is called the origin, the horizontal line the x axis, the vertical line the y axis. The x distances, measured to the right or left of the y axis, are called abscissae, the y distances,

measured above or below the x axis, are called ordinates; distances measured upwards and to the right from the origin are considered to be positive, and those measured to the left and downwards from the origin are negative. (This is merely a convention.) Suitable scales are chosen for both x and y axes so that the plotted points or the final graph comfortably fill the paper. Considerable judgment, experience or even rapid calculation are necessary when good graphs are to be drawn to find suitable scales and suitable positions for the axes. When a point is plotted its co-ordinates are given in the order x (abscissa) and then y (ordinate). Thus the point 3.5. means a point 3 units to the right of the y axis and 5 above the x axis. The origin (the intersection) of the axes is the point 0.0. It should be seen that the x axis represents a line for which all y values are zero, i.e., its equation is $y = 0$.



Similarly the y axis is the line $x = 0$.

Practice should be given in plotting points, in joining up the points to form straight lines, squares, rectangles and triangles. If areas of simple rectilinear figures are found by counting squares, care should be taken to find the area which is represented by a single large square, for often the scale along the x axis is not equal to that along the y axis.

The straight line graph through the origin represents simple direct proportion. The simple expression is $y = ax$, and it is sometimes written $y \propto x$. There are many other cases of straight line graphs which do not go through the origin; for instance: graphs to show the conversion of Fahrenheit to Centigrade scales; the total cost per quarter of electricity on the 'fixed-charged scale'; the total cost of a telephone (rent and calls); certain income-tax, postage, cost of telegram problems, etc.

Here the equation is $y = ax + b$, which is the general expression for a straight line and is called a linear equation. In the particular case of the thermometric scales we may write it:

$$F = \frac{9}{5} C + 32$$

(F = temperature on the Fahrenheit scale and C = temperature on the Centigrade scale.) As two points are sufficient to determine a straight line, it is only necessary to have two sets of values of x and y (co-ordinates) in order to draw the graph of a linear equation.

This leads us to the solution of simultaneous linear equations. Although these have not the practical importance which has been given them in the past, they illustrate a useful principle. Every point on the straight line representing the equation $y = ax + b$ gives a pair of values for x and y . Now if another straight line $y = cx + d$ is drawn on the same graph, the point at which the two lines cross

gives values of x and y which are common to both equations.

The following is a typical example where a number of straight lines may be drawn on a single graph.

A householder has a choice of three ways of buying electricity:

a. To pay $4\frac{1}{2}$ d. per unit.

b. To pay £1 each quarter and 1d. for every unit which he uses.

c. To pay £1 10s. od. each quarter and $\frac{3}{4}$ d. for every unit which he uses.

Graphs will show which terms are the most advantageous when various numbers of units are consumed. They will also show just at what point one way of buying electricity is as economical as another.

Inverse Variation

The equation of simple inverse variation is $y = \frac{x}{k}$ or $xy = k$.

This may be used to illustrate the common arithmetical examples of inverse proportion. The curve is a 'rectangular hyperbola' and is well known through its association with Boyle's law; but actually Boyle's law is only a first approximation to the real law of variation of the pressure and volume of a gas, and when pressures are high and temperatures are low the 'law' is not applicable. More useful examples of inverse variation will be found in cases where a quantity, which is a product of two other varying quantities, remains constant, e.g., a rectangle of given area but variable sides; a fixed amount of work done by different numbers of men working corresponding numbers of days (e.g., more men, fewer days, but this must not be pushed to unreasonable limits for even 10,000,000 men could not build a battleship in 1 minute); the number of hours taken by electric bulbs of different wattage to consume 1 unit of electricity; the

quantity of a commodity the price of which fluctuates, which may be bought for a fixed sum; wave-length and frequency; amperes and volts for constant wattage, etc.; the number of revolutions per minute of the wheel of a vehicle travelling at a certain speed and the diameter of the wheel, etc. It will be observed that the rectangle contained by the co-ordinates of any point on the graph always has the same area.

Variation with the 'direct square'

The graph of $y = x^2$.

For this graph, which is called a parabola, a smaller scale will be necessary for the y values than for those of x , as the graph should fill the paper comfortably. The axis of x is drawn at the bottom of the paper and the axis of y down the middle. If the graph is plotted with both positive and negative values of x the curve will be symmetrical about the axis of y . The graph may be used to find squares and square roots, surfaces in relation to distances, etc.; and later we shall meet similar curves, though in different positions with respect to the axes, when solving quadratic equations.

The parabolic curve is often found when considering motion under gravity neglecting air resistance. However, it must be stated that air resistance upsets our calculations much more than is generally supposed. (A stone dropped from a church steeple through 200 feet usually finishes the last part of its journey at a constant velocity; and the angle at which the projectile of a high-velocity gun, fired at a range of 2 miles, strikes the ground is considerably greater than the angle of elevation of the gun.) A boy running horizontally and then jumping into water describes half a parabola; a horizontal jet of water issuing from a nozzle turns to the ground following a parabolic curve. The reason is easily seen. In a given time, a body projected horizontally travels in this direction a distance proportional to time, but

it is also falling under gravity a distance proportional to the square of time. The resultant movement of the body is compounded from these horizontal and vertical motions.

Averages and Rate

The idea of average may be approached by considering cricket averages, the average height of boys of a particular age or of men, the average mark of the class in a test, etc. The idea of averages and 'means' becomes more important every year as arithmetical methods are now applied to many social and economic matters in addition to engineering and the exact sciences; and even in the latter, statistical methods are becoming more usual. In the Senior School it will suffice to deal with averages by adding the quantities together and dividing by the number of things. The principle may be demonstrated by dealing with a number of results of a scientific experiment, the speeds of a tram or a motor-car, the lengths of sticks in a bundle, the daily yield of a cow in milk. (It must be understood that this is not the only way of taking averages.)

The desirability of taking as many observations as possible should be seen. The work should be carefully performed for it will pave the way to investigations in later years on the 'curve of error', standard deviations, probability, etc., which may be studied by more intelligent children in technical colleges at a later date. Here there is every opportunity to make the matter an attractive one, for there is something very attractive about the possibility of applying mathematics to large numbers of things which differ in some respects.

A graph showing the speed of a motor-car in going a journey of 100 miles. Time is plotted horizontally and the speed vertically. The start, the effects of a piece of rough road, a congestion of traffic, accelerating to pass another car, a '30 mile' an hour limit, a stretch of good road, a stop for

tea, climbing a hill, etc., may all be shown on the graph. A more difficult graph might even show the effect of a mistaken direction and going back to find the correct road. (Here the x axis would be drawn across the middle of the paper.)

The graph should be carefully examined and reasons sought for its appearance at every point. It will be appreciated that if two vertical lines are drawn 'enclosing' a definite interval of time the area enclosed by these lines and the lines cut off by them on the x axis and the graph will represent the distance travelled by the car during that interval of time. Speed (if constant) = $\frac{\text{Distance}}{\text{Time}}$ (using suitable units).

Thus Distance = speed $\times$ time.

If a sufficiently narrow time strip is taken we may suppose that the speed does not materially alter during the small interval of time. The strip, which may be regarded as a long rectangle, represents the distance travelled. The area bounded by the 'graph' and the x axis may be considered to be composed of a great number of narrow vertical strips and it represents the total distance travelled between starting and stopping; or again, by drawing vertical lines to represent certain times, the distance travelled in a given interval. If the graph goes below the x axis the area so made will represent a part of the *total* distance travelled but the question of the direction of travelling will have to be considered when finding the distance travelled in a certain direction.

When the car is going at a steady 30 miles an hour (e.g., when going through a 'built-up' area) or, indeed, at any constant speed, the graph will be seen to be a straight line parallel to the axis of x , e.g., the graph does not slope or show any change of direction when the car is travelling at a uniform speed or is at rest.

When the graph slopes upward the car is changing its speed and the greater the bend in the graph at any particular point the more is the speed of the car changing at that instant. Similarly if the car is 'slowing down' the graph turns downward in proportion to the rate of its change of speed. The rate of change of speed is called acceleration or retardation as the case may be; but it is usual to think of a retardation as a negative acceleration. Thus, it is not difficult to see that acceleration is given by the amount of slope of the graph (the gradient) at any point. With curves we determine the slope by drawing a tangent to the curve at the required point. To do this properly is more advanced work, but a general idea may be obtained by 'common sense' drawing with a ruler.

The average speed throughout the whole or a portion of the journey may be found by finding the total distance travelled by counting the squares in the area enclosed by the graph, the x axis and the two time lines, and dividing by the distance between the time lines which represents the total time of the journey. The average speed may be represented by drawing a suitable straight line parallel to the x axis. This method of finding averages may be widely applied. (*N.B.* In counting squares, when less than half a small square is left within the graph it may be neglected, more than half a small square being counted as a unit. When interpreting the total area of the squares it is necessary to keep in mind the x and y scales.)

Such graphs may be widely applied in dynamics and they lead the way to the calculus in later years. The graph of a body falling freely under gravity is another useful case and neglecting air resistance (which in practice can never be done) we can also obtain a simple algebraic formula. The acceleration due to gravity is constant and therefore the slope of the velocity-time graph is also constant, i.e., it is a straight line drawn through the origin of the graph (pro-

vided that the body starts from rest), and making an angle with the x axis according to the scale chosen. The speed after any interval of time may be read directly from the graph, and the distance travelled is given by the area enclosed by the graph the x axis and the time ordinates.

If the body starts with an initial velocity (u), the straight-line graph will cut the axis of y at a distance u above the origin. The velocity v at any time is seen to be $v = u + at$ (' a ' being the acceleration or gradient of the line).

The distance travelled during this time is equal to the area of a rectangle (ut) and a triangle ($\frac{1}{2} at \times t = \frac{1}{2} at^2$).

$$\text{Distance travelled} = ut + \frac{1}{2}at^2.$$

A simple algebraic manipulation where t is removed from this equation by substituting its value $t = \frac{v - u}{a}$ gives the third (but not independent) equation: $v^2 - u^2 = 2as$ (s = distance travelled) which is useful for calculations where time does not enter.

It will be noted that:

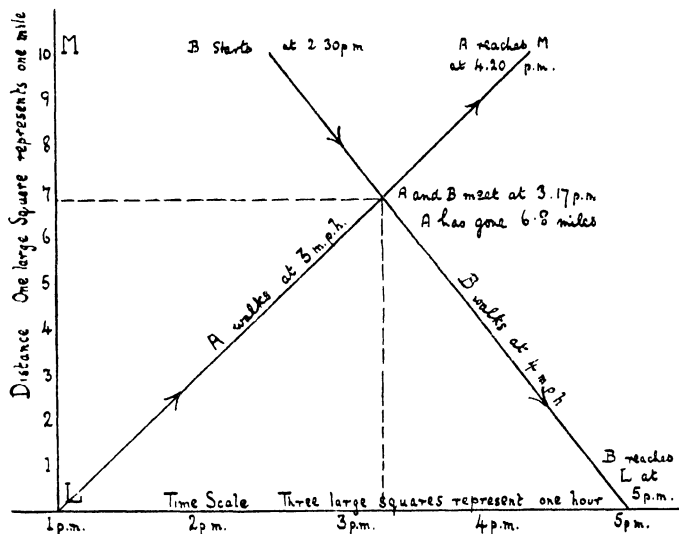
Speed (or velocity) is the rate of change of position.

Acceleration is the rate of change of velocity.

Graphs may be drawn to solve problems concerning distances, speeds, and time where acceleration does not enter and time is plotted against distance, e.g.: Problems concerning trains overtaking one another, men walking in opposite directions over the same route at different speeds and meeting, etc. Here the speed is given by the slope of the graph; and the speeds and other directed quantities must be given a correct sign.

Example. A starts at 1 p.m. from L to walk to M, a distance of 10 miles, at 3 miles an hour. B starts at 2.30 p.m. to walk from M to L at 4 miles an hour. When and where will they meet?

It is useful in simple ideal cases to follow the processes of graphical working by their algebraic equivalents.



The Solutions of Equations by Graphical Methods

The solution of simultaneous equations of two unknowns, already referred to, is not a matter of any difficulty for a graph drawn on a plane surface is seen to concern itself with x and y values, and the point of intersection of the two straight lines clearly represents values of x and y which are common to both equations. Children who will solve such equations readily may often fail to understand how a quadratic equation can be solved by a graph. Suppose the equation is $ax^2 + bx + c = 0$. This contains no mention of 'y' and to introduce another symbol seems an unnecessary step. The matter may be explained as follows. All equations are solved graphically by finding the point or points where one line cuts another. In the case of an equation where a function of x is equated to 0, we may first regard it as being

equated to y , which is then given the value 0. Thus $ax^2 + bx + c = 0$ reads

$$y = ax^2 + bx + c$$

$$y = 0.$$

and having plotted the equation $y = ax^2 + bx + c$, we find where it cuts the line $y = 0$, i.e., the axis of x . Again the equation $ax^2 + bx + c = 0$ may be written:

$$x^2 = -\frac{bx}{a} - \frac{c}{a} = y \text{ (suppose)}$$

$$\text{or } y = x^2$$

$bx + ay + c = 0$, and thus the quadratic equation may be solved, by drawing the curve $y = x^2$ (a parabola through the origin) and the straight line $bx + ay + c = 0$ and finding the co-ordinates of the points where they cut. If equations of a higher power of x arise in the child's experience, graph-drawing (and factorizing if this is possible) are the only possible methods of solution which may be used at this stage.

In the actual drawing of the graph the following points should be remembered. Neat columns should be made to show the corresponding values of the independent variable (x) and the dependent variable (y). The curves produced by simple algebraic functions are always 'shapely', and the eye should detect small divergencies from the correct form. At turning points, and at all places on the graph where there is the slightest possibility that the exact position of the line may be unknown, additional values of x and y must be found.

As the work with graphs proceeds, an examination of the equation or function to be graphed should give a general idea of the type of curve and its position—even before the first point is placed on the graph-paper.

The standard of neatness in formal graphical work should be equal to that in mechanical drawing; but much useful

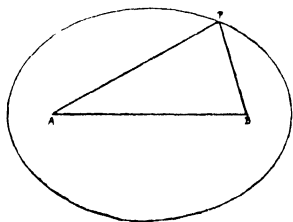
demonstrative work may be done by drawing axes on the blackboard or on a piece of plain paper and sketching roughly the general shape of the graph.

Although no formal algebraic geometry is taken in Senior Schools, the older and more intelligent children may compare algebraic expressions and their graphs. In the case of a straight line they can investigate the effect on the equation by altering the angle which the line makes with the x axis, or again the effect of 'translating' the line without altering its direction.

The equation of a circle with its centre at the origin is useful and is easily understood. As all points on the circumference of the circle are at the same distance from the origin we have by Pythagoras' Theorem $x^2 + y^2 = r^2$ (if x and y are the co-ordinates of the point and r is the radius of the circle).

The parabola has already been examined in curves of the type $y = x^2$. All parabolas have the same shape. Simple accounts of parabolic reflectors, the application of the curve to astronomy, and quick ways of drawing it may be mentioned.

The elliptical shape is of such wide application that a few experiments may be performed to show its nature. The word 'oval' (egg-shaped) is sometimes loosely used for 'ellipse',



but eggs are not strictly elliptical in section. In practice, the ellipse is used in design and architecture, in engineering, in astronomy and for its focal properties, etc. It may be demonstrated by drawing a circle on a piece of rubber or other

material, which is then pulled in one direction. The circle is seen to be a particular case of an ellipse. Ellipses may be drawn by placing two pins vertically in a piece of paper at a distance of (say) 3 inches apart, knotting a ring of thread of total length (say) 8 inches, putting this round the pins, and keeping it taut with a pencil point while drawing the curve. The points A and B are called foci and if P is a point on the ellipse $PA + PB$ is a constant.

Children will be interested in a *descriptive* account of conic sections since the time of the Greeks. A cone may be made of clay or plasticine, and cut in various planes to illustrate the parabola, ellipse and hyperbola.

Some suggestions for further graphs:

The period of swing of a pendulum plotted against its length; a cooling curve, and its applications; a calibration curve for a thermometer. The straight line graph of Hooke's Law—the amount of extension of a spring (or a piece of rubber) plotted against the weight fastened to the end of it. After the graph has been drawn it will be interesting to discuss the limitations of the law.

The prices of commodities fluctuating over a period of years. Discuss possible reasons.

Sine and tangent graphs and their uses. (See section on Trigonometry.)

Graphs showing national and local expenditure, birth and death rates, and other statistics. Many publications containing interesting statistics which may be converted to graphs are now available, e.g., the *Statistical Year Book of the League of Nations* (Allen and Unwin); *Whitaker's Almanack*, etc. Children will search for data, plot a graph, interpret it, and use it subsequently.

Graphs of powers of 2 and 10 as an introduction to logarithms.

Graphs to show Compound Interest and other money problems.

Quadratic Equations

Many problems in physics, mechanics, geometry and arithmetic resolve themselves into quadratic equations. These must be regarded as work for the senior forms, and

it is probable that they will be outside the scope of the work of most senior schools.

Quadratic equations may be approached by factorizing, graphically, and then by completing the square; and historic examples may always be found.

Quadratic Equations

1. By factorization.

$$\begin{aligned} \text{e.g. } x^2 + 7x + 12 &= 0 \\ (x + 3)(x + 4) &= 0 \\ x + 3 &= 0 \text{ or } x + 4 = 0 \\ x &= -3 \text{ or } -4 \end{aligned}$$

$$\begin{aligned} 8x^2 - 26x + 15 &= 0 \\ (4x - 3)(2x - 5) &= 0 \\ 4x - 3 &= 0 \text{ or } 2x - 5 = 0 \\ 4x &= 3 \text{ or } 2x = 5 \\ x &= \frac{3}{4} \text{ or } 2\frac{1}{2}. \end{aligned}$$

This method is obviously only applicable to cases where it is possible to obtain factors.

2. By completing the square.

x^2	$\frac{7}{2}x^2$
$\frac{7}{2}x$	$\left(\frac{7}{2}\right)^2 = \frac{49}{4}$

$x^2 + 7x + 12 = 0$
 $x^2 + 7x + \frac{49}{4} = \frac{49}{4} - 12$
 $\left(x + \frac{7}{2}\right)^2 = \frac{49 - 48}{4} = \frac{1}{4}$
 $x + \frac{7}{2} = \pm \sqrt{\frac{1}{4}} = \pm \frac{1}{2}$
 $x = -\frac{7}{2} + \frac{1}{2} = -3$
 or $-\frac{7}{2} - \frac{1}{2} = -4.$

Diagram to illustrate the completion of the square

Divide through by the coefficient of x^2 if this is not already unity. Remove the term not containing x to the right-hand side, add the square of half the coefficient of x to both sides. The left side is now a square. Take the square root of both sides and notice that (when algebraically considered) the square root of any number must take both + and - signs.

$$\begin{aligned} \text{e.g. } (+7) \times (+7) &= 49 \\ (-7) \times (-7) &= 49 \\ \therefore \sqrt{49} &= +7 \text{ or } -7 \end{aligned}$$

The method of completing the square is really only a way of finding factors for an algebraic expression of the form $x^2 + ax + b$. The factors, of course, do not usually contain simple integral numbers such as in case (1). Above we give a simple example to illustrate the principle by means of a diagram, but normally this example would be worked by factorization.

The equation $x^2 + bx + c$ may be written $x^2 + bx = -c$.

Now an L shaped diagram may be drawn representing $x^2 + bx$ as a square of side x with two rectangular arms each $\frac{b}{2}$ long and x wide. To complete the square we must add a square of side $\frac{b}{2}$ or $\frac{b^2}{4}$ in area, i.e., to complete the square we take half the coefficient of x and square it. (N.B. Coefficient of x^2 is unity or is made into unity by dividing.) The principle may be illustrated by squaring $(x + 1)$, $(x + 2)$, $(x + \frac{1}{2})$, $(x + 4)$, etc., and by completing $x^2 + x$, $x^2 + 2x$, $x^2 + 4x$, $x^2 + 3x$, etc.

The General Solution of a Quadratic Equation

$$ax^2 + bx + c = 0$$

Divide by a ,
$$x^2 + \frac{bx}{a} = -\frac{c}{a}$$

Complete the square, $x^2 + \frac{bx}{a} + \frac{b^2}{4a^2} = \frac{b^2}{4a^2} - \frac{c}{a}$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

Take the square
root of both sides

$$x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The general solution of a quadratic equation is rather beyond the scope of Senior School work, but it is important, as it throws a light on the nature of quadratic equations in general; for instance, it will be seen that if $b^2 - 4ac = 0$, i.e., $b^2 = 4ac$ the two solutions will be equal, or if b^2 is less than $4ac$ there will be no real solutions (for the square root of a negative quantity has no reality). The parabolic graph of a quadratic equation cuts the axis of x in two places if there are two real roots, touches it if the roots are equal and does not touch or cut it if there are no real roots.

Logarithms

If logarithms are to be included in Central School work (as they will in A and probably in B classes) they should be taught as early as is convenient, and subsequently used for computation whenever the need arises.

Although it is hoped that many of the most intelligent children will be sufficiently interested in later years to investigate logarithms from the point of view of the exponential function and the law of growth, it will have to suffice at school to develop them from the simple index laws, and use them as a means of simplifying calculation. The multiplication of two large numbers is a much more difficult and tedious process than adding them together;

and thus a method of multiplication by merely adding two numbers (which are found from the original numbers by means of tables) results in a saving of time and paper.

John Napier (1550–1617), Baron of Merchiston in Scotland, revealed his use of logarithms in 1614. The following passage from his work, *Mirifici logarithmorum canonis descriptio*, is interesting:

‘Seeing there is nothing (right well beloved students in Mathematics) that is so troublesome to mathematical practice, nor that doth more molest and hinder calculation than the multiplications, divisions, square and cubical extraction of great numbers, which, besides the tedious expense of time, are for the most part subject to many slipping errors, . . . and having thought upon many things I found at length some excellent brief rules. But amongst all, none more profitable than this, which, together with the hard and tedious multiplications, divisions and extractions of rootes, doth also cast away from the work itselfe even the very numbers themselves that are to be multiplied, divided and resolved into rootes, and putteth other numbers in their place, which perform as much as they can do, only by addition and subtraction, division by 2 division by 3; which secret invention being (as all other good things are) so much the better as it shall be the more common. I thought good heretofore to set forth in Latine for the publique use of mathematicians. But now some of our countrymen in this island well affected to these studies and the more publique good procured a most learned matnematician to translate the same into our vulgar English tongue, who, after he had finished it, sent the cobby of it to me to be seen and considered on by myself.’

Napier’s achievement is the more remarkable in that his discovery was based on sines and velocities and not

on indices.¹ His work was improved by his contemporary, Henry Briggs (1556–1631), of Gresham College. Logarithms became very popular for they were championed by Kepler, who in common with other astronomers saw how they simplified the cumbrous and lengthy calculations which were entering astronomy and navigation. The connection between logarithms and exponentials appeared much later; and the tables of Napier, Briggs and Adrian Vlacq (a Flemish bookseller who lived in London for some years and was bitterly disliked by the poet Milton) seem very fearsome when compared with our simple system. The word ‘logarithm’ means ‘ratio-number’. Work may begin by a consideration of the powers of 2. A graph may be drawn of $2^1, 2^2, 2^3, 2^4, 2^5, 2^6$ plotted against the indices 1, 2, 3, 4, 5, 6. It is necessary to assume that some knowledge of indices has already been acquired. By demonstration it can be shown that for positive integers $x^5 \times x^3 = x^8$ and $x^5 \div x^3 = x^{5-3} = x^2$ or in the general case $x^m \times x^n = x^{m+n}$ and $x^m \div x^n = x^{m-n}$ and we see the principle of multiplying the terms by adding the indices and dividing by subtracting them. We have no right to assume that what has been demonstrated for positive integral indices is also true for fractional and negative indices. From our simple graph it would appear that we might interpolate and extrapolate values in the graph and find an interpretation for fractional and negative indices. If we make the assumption that the index-laws are true for all indices we are able to give meanings for zero, negative and fractional indices, which will yield a consistent set of results.

To find the meaning of x^0 , multiply it by x^m which yields x^{m+0} or x^m , i.e., the multiplier is equal to the product, and therefore the multiplicand must be interpreted as being equal to unity.

¹Napier's priority of claim has not remained undisputed. He was anticipated in some ways by mathematicians such as Bürgi, working abroad.

To find the value of x^{-m} multiply it by x^m

$$x^{-m} \times x^m = x^0 = 1$$

$$\therefore x^{-m} = \frac{1}{x^m} \quad \text{e.g. } x^{-2} = \frac{1}{x^2}$$

The idea of the extension (subject to certain assumptions) of our simple positive integral indices should be stressed.

Further index laws: $(x^m)^n = x^{mn}$

$$\text{and } \sqrt[n]{x^m} = (x^m)^{\frac{1}{n}} = x^{\frac{m}{n}}$$

or in words, 'to raise to a power multiply the index by the number of power', and 'to find a root divide the index by the "number" of the root', e.g., 2 for a square root, 3 for cube root, etc.

Base	Logarithm or Index	Number
2	- 3	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$
2	- 2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$
2	- 1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$
2	0	$2^0 = 1$
2	1	$2^1 = 2$
2	2	$2^2 = 4$
2	3	$2^3 = 8$
2	4	$2^4 = 16$
2	5	$2^5 = 32$
2	6	$2^6 = 64$

From our graph we can obtain the logarithms (the indices) to base 2 of numbers between 2 and 64. If we have extended our graph to include 2^0 (1), 2^{-1} ($\frac{1}{2}$), 2^{-2} ($\frac{1}{4}$) and 2^{-3} ($\frac{1}{8}$) we can proceed to construct the outline of a table of logarithms to base 2. It will be seen that if instead of the numbers themselves, we think of the indices of the powers of two which produce these numbers, we can perform operations of multiplying, dividing, raising to a power and 'extracting a root' by adding, subtracting, multiplying and dividing indices respectively, and then taking the final index (or logarithm) and finding a number which corresponds to it.

It will probably be found worth while to use 2 as the base for a preliminary explanation, but it will be appreciated that as our decimal system already uses powers of 10, this will generally be the most useful base for ordinary purposes of computation. Nevertheless, intelligent children who are likely to use logarithms later from an analytical point of view should be told that the 'natural' base is neither 2 nor 10 but a number which has the utmost significance in later mathematics and its applications to physics, engineering, probability, statistics, etc.

A graph may be plotted of powers of 10 against the numbers of the corresponding indices. This is not so satisfactory as the graph of the powers of 2, as powers of 10 soon yield large numbers which cannot well be included on the graph paper, without reducing the scale to such an extent that the graph becomes useless. Also a table of logarithms to base 10 will be constructed as follows on the top of next page.

For numbers between 1 and 10 our graph would lead us to believe that there is a corresponding series of logarithms between 0 and 1. If we make a table of these from a good graph we are then able to write down the logarithms of any number we please within certain limits of accuracy. Rough estimates of logarithms may be found either from the graph

Index (Logarithm)	Number	
— 4	10^{-4}	$\frac{1}{10,000} = \cdot 0001$
— 3	10^{-3}	$\frac{1}{1,000} = \cdot 001$
— 2	10^{-2}	$\frac{1}{100} = \cdot 01$
— 1	10^{-1}	$\frac{1}{10} = \cdot 1$
0	10^0	1
1	10^1	10
2	10^2	100
3	10^3	1,000
4	10^4	10,000

or by simple experiments with numbers, e.g. to find $\log_{10} 2$ approximately:

$$2^{10} = 1024 \quad 10^3 = 1000$$

$$\therefore 2^{10} = 10^3 \text{ approximately.}$$

$$\therefore 2 = 10^{\frac{3}{10}} \text{ (taking the 10th root of both sides).}$$

$$\therefore \log_{10} 2 = \frac{3}{10} \text{ approximately}$$

(the result given in four-figure log tables is $\cdot 3010$) or again, to find $\log_{10} 3$ very roughly:

$$3^2 = 9 \text{ (call it } 10, \text{ and notice the error)}$$

$$3^2 = 10 \text{ (approx.)}$$

$$\therefore 3 = 10^{\frac{1}{2}} \text{ (approx.)}$$

$$\therefore \log 3 = \frac{1}{2} \text{ (approx.)}. \text{ The correct answer to 4 places is } \cdot 4771.$$

A nearer approximation is given by considering 3^9 which is approximately 20,000 or $2 \times 10,000$.

Again, $7^4 = 2401 = 3 \times 8 \times 10^2$ (approx.) $\therefore 4 \log 7 = \log 3 + 3 \log 2 + 2$, etc

Children will be interested if given $\log_{10} 2$ and $\log_{10} 3$ from the table book (or as estimated above) and asked to calculate

$$\log 4. (4 = 2^2), \log 5. (5 = \frac{10}{2}), \log 6. (6 = 3 \times 2),$$

$$\log 8. (8 = 2^3), \log 9. (9 = 3^2), \log 25. (25 = \frac{100}{2^2}), \text{ etc.}$$

and many others. The results may be checked by referring to the tables. This should be combined with practice in using tables quickly, accurately and confidently.

Suppose we need the logarithm of 5912. We can regard it as 5.912×1000 , look up the logarithm of 5.912 and add to this *logarithm*, the logarithm of 1000 (3); or again, if we want the logarithm of .005912 we may regard it as $5.912 \div 1000$, i.e., in index form $5.912 \div 10^3$ or 5.912×10^{-3} . We therefore look up the logarithm of 5.912 and add to it the logarithm of 10^{-3} , or -3 . It is convenient, as will appear when examples are worked to regard this logarithm as a negative integer + a decimal fraction. In order to preserve uniformity of method, we set down this type of logarithm as 3.7717. (If set down as -3.7717 the decimal fraction would also be negative.) 3 is the logarithm of 10^{-3} ($\frac{1}{1000}$) and is an integer, called the *characteristic*. The decimal fraction in the logarithm is called the *mantissa*. When we have a logarithm and require the number corresponding to it, the mantissa only is found from the table, whilst the characteristic tells us where to put the decimal point.

From the table given above it is seen that numbers between 1 and 10 have no characteristic; between 10 and 100 a characteristic of 1, between 100 and 1000 a characteristic of 2; between .1 and 1 a characteristic of -1 (written $\bar{1}$), between .01 and .1 a characteristic of $\bar{2}$ and so on. Common sense should therefore tell us the number of the characteristic, and when going from logarithm to

number, where to put the decimal point; but the following rule or its equivalent is sometimes used. 'The characteristic if positive is one less than the number of digits to the left of the decimal point, and if negative one more than the number of "noughts" immediately after it', e.g.

Number	Log
5.912	.7717
5912	3.7717
.005912	$\bar{3}.7717$
591200	5.7717
.5912	$\bar{1}.7717$

Although mechanical skill in using tables and in manipulating logarithms is looked for, some general work intended to show their nature is necessary. The child should have a clear grasp of the fact that a logarithm is an index and that, for instance, the expressions $y = \log_{10} x$ and $10^y = x$ are only two ways of saying the same thing.

Several methods are in vogue for setting out problems involving logarithmic working.

(1) *The Index Method*

Here the numbers are expressed as powers of 10 throughout.

Example. Evaluate $\frac{(489.2)^2 \times .0003317}{(19.82)^3}$

$$\begin{aligned}
 & \frac{(489.2)^2 \times .0003317}{(19.82)^3} \\
 &= 10^2 \times \log 489.2 \times 10^{\log .0003317} \div 10^{3 \times \log 19.82} \\
 &= 10^2 \times 2.6895 \times 10^{\bar{4}.5207} \div 10^{3 \times 1.2971} \\
 &= 10^{5.3790 + \bar{4}.5207 - 3.8913} \\
 &= 10^{\bar{2}.0084} \\
 &= .01020
 \end{aligned}$$

This method tends to prevent the troubles which arise

from carelessly allowing numbers and logarithms to become mixed, or the absurdity of equating a number and its logarithm (e.g., $2 \cdot 3010 = 200$). It also prevents adding and subtracting by logarithms—a fault which is sometimes seen in early attempts. It has the great disadvantage that it involves writing extra figures and that the figures in the index have to be written very small and extended so far that they tend to fall away from the 10 to which they are the index.

(2) *The Column Method*

Here the expression to be worked out is set down and examined (and any additions or subtractions dealt with before beginning to use logarithms). All the numbers involved are written down in a column, against them in a second column their respective logarithms, and in a third column the operations of adding and subtracting, etc., the logarithms are performed. This, or an adaptation of it, is probably the most general method where logarithms are used for general computation in the applied sciences, *and is recommended here.*

$$\frac{(489 \cdot 2)^2 \times \cdot 0003317}{(19 \cdot 82)^3}$$

$$= \cdot 01020$$

[Rough estimate.

$$\frac{(500)^2 \times \cdot 00033}{(20)^3}$$

$$= \frac{1}{10} = \cdot 01]$$

Number	Logarithm	
$(489 \cdot 2)^2$	$2 \times 2 \cdot 6895$	5.3790
$\cdot 0003317$		4.5207
$(19 \cdot 82)^3$	$3 \times 1 \cdot 2971$	1.8997
$\cdot 01020$		3.8913
		2.0084

(It is unlikely that such problems will be found in the child's experience, but the above example is chosen to reveal certain points in the mechanical methods employed in using logarithms. Mensura-

tion, surveying and science will supply simpler and more useful examples.)

(3) *The Equation Method*

This is perhaps rather a pedantic method and it involves the use of an unknown, where one could be avoided.

$$\text{Evaluate } \frac{(489.2)^2 \times .0003317}{(19.82)^3}$$

$$\text{Let } x = \frac{(489.2)^2 \times .0003317}{(19.82)^3}$$

$$\begin{aligned}\log x &= 2 \log 489.2 + \log .0003317 - 3 \log 19.82 \\ &= 2 \times 2.6895 + 4.5207 - 3 \times 1.2971 \\ &= 5.3790 + 4.5207 - 3.8913 \\ &= 2.0084\end{aligned}$$

$$\begin{aligned}\therefore x &= \text{antilog } 2.0084 \\ &= .01020\end{aligned}$$

$$\text{Ans. } .01020$$

When more complex expressions containing + and - signs are to be evaluated, care should be taken to work separately the several parts by logarithms and perform addition and subtraction in the normal way. The young student is sometimes tempted to perform addition and subtraction by logarithmic means.

Practice should be obtained in finding squares and square roots quickly by means of logarithm tables.

Negative characteristics are apt to give some trouble at the start, but if it is remembered that the mantissa must be kept positive a little ingenuity will serve to overcome all difficulties. To think of a logarithm such as $\bar{3}.6982$ as $-3 + .6982$ may help at first, and teachers may use a coloured chalk for writing down the characteristics. The following example is not a practical one but it reveals certain difficulties in the use of logarithms, and for that purpose alone it is included here.

$$\text{Evaluate } \frac{1}{(\cdot 0004687)^{\frac{3}{7}}}$$

$$\log \frac{1}{(\cdot 0004687)^{\frac{3}{7}}} = \log 1 - \frac{3}{7} \times \log \cdot 0004687$$

$$\begin{aligned} &= 0 - \frac{3}{7} \times 4.6709 \\ &= 0 - \overline{10} \cdot 0127 \div 7 \\ &\quad (\overline{12} + 2 \cdot 0127) \\ &= 0 - \frac{14 \cdot + 4 \cdot 0127}{7} \end{aligned} \left. \begin{array}{l} \text{Observe method of} \\ \text{multiplication and} \\ \text{division. Normally} \\ \text{these stages will not} \\ \text{be written down.} \end{array} \right\}$$

$$\begin{aligned} &= 0 - 2.5732 \\ &= 1.4268 \end{aligned}$$

(The subtraction from 0 will give no difficulty if it is performed algebraically.)

$$\begin{aligned} \therefore \text{Required number} &= \text{antilog } 1.4268 \\ &= 26.72 \end{aligned}$$

Some General Remarks on the Teaching of Logarithms

(1) Many teachers prefer to confine all work to logarithm tables only, and to refer back to these tables for looking up antilogarithms. This prevents children from looking up logarithms in antilogarithm tables and vice versa. Many examining bodies give tables of logarithms only (and, of course, 'anti-' or arc-sine tables are not used in schools). Practice should be given in finding the logarithms of numbers and numbers corresponding to logarithms from the same sheet of tables.

(2) At first, all mantissae should be kept to four figures (if four-figure tables are used), even if this means adding 0's.

(3) Examples should be obtained from books used by engineers, and other works which contain formulae.

(4) At a later stage, compound interest may be developed to give a simple idea of the growth function by using logarithms.

(5) As a logarithm is an index, logarithms offer a way of solving certain simple equations where the unknown is an index. (See Section on Compound Interest.)

(6) The simple processes of multiplying and dividing by logarithms become more real if a slide-rule is demonstrated. A simple instrument may be constructed by gumming two slips of logarithmic paper to two flat pieces of wood kept together by elastic bands, so that the pieces of logarithmic paper can be made to slide one against the other. A simple slide-rule made of cardboard may be bought for a few pence.

(7) In later technical work, when a formula is required to express the relation between certain varying quantities, the advantages often gained by plotting graphs logarithmically should be shown.

CHAPTER IX

GEOMETRY AND MENSURATION

IN both English and German Schools, the last quarter of a century has seen the 'flight from Euclid'. The reaction has been so great that there is now a tendency to drive all deductive geometry from Secondary Schools, but if it is properly taught, the niche that is occupied by this branch of the subject can hardly be filled by any other study.

Formerly, the teaching of geometry often failed for some of the following reasons:

(1) It was left unconnected to other branches of mathematics and its applications were neglected.

(2) It was not approached properly through experimental and inductive geometry. Deductive work was started at too early a stage, and the nature of the logical processes used and the unity of the system were not grasped.

(3) It tended to degenerate into a series of 'dodges' related only to the immediate theorem.

(4) The stress was put on 'bookwork theorems' which were learnt by rote. Riders were neglected, through lack of time and because they were considered to entail a dangerous risk at examinations; safe 'analytical' methods were often preferred to deductive methods which might absorb much time but give little result.

(5) Children were not in sympathy with Euclid's methods, as they had anticipated the results by experiment or intuition. 'Why should we spend time proving by a tedious method that two sides of a triangle are greater than a third, for even animals realize this; and, in any case, it seems to

follow from the fact a straight line is the shortest distance between two points?’

With older children, an endeavour should be made to explain (a) the value of deductive geometry in the mathematics scheme and its relation to other mathematical studies. (b) The nature of deduction, and why training in deductive thinking is useful. (c) Why the Greeks prized deductive thinking, and to give some account of Greek mathematics.

In Central Schools, however, the problems raised by the teaching of deductive geometry hardly arise, owing to the short time available for mathematics and the low leaving-age. The teaching of Geometry should develop through Intuition, Induction (Experiment), and Deduction; but in Central Schools the first two stages will occupy most of the time available although simple deductive processes may sometimes be introduced in the last year as a means of summarizing the work in precise and logical form.

The stress will be placed on the applications of simple geometrical tools to handicraft, drawing, surveying, projections; and every opportunity will be given for children to handle and measure solid objects to co-ordinate hand and eye, and extend their knowledge of spatial relations. In the past elementary geometry has overlooked the three-dimensional aspects of the subject, which from the practical point of view are most important.

Angles

(1) How an angle is made—by opening a door, a book, a pair of scissors.

(2) A round of angles adds up to 4 right-angles or 360° .

(3) A ‘straight’-angle is two right-angles.

(4) How to make a right-angle by folding paper. Other angles made by folding paper. How to make angles of 60° and 30° by drawing equilateral and other triangles.

Set-square angles. Make angles larger than 90° , then larger than 270° . (Rotate a wheel slowly noticing one of the spokes.) Use of a protractor for measuring angles. (This is attended with some difficulty at first for children find difficulty in holding the instrument properly and need constant practice in reading it in the correct direction.)

(5) The properties of angles made by opening a pair of tongs, or two Meccano strips loosely hinged together by a single bolt in their middle (vertically opposite angles are equal). Angles made when a line crosses two parallel straight lines. This may be shown with jointed laths, Meccano strips, 'crossings' on model railway-lines, etc.

(6) How we measure the angle between two plane surfaces or between a line and a plane surface. Use of T-square in drawing and other instruments in craft-work.

Lines and Surfaces:

What is a straight line?

How a gardener makes a straight line. Tests for the straightness of a piece of wood or metal by looking along it. How to draw a straight line by using a ruler. How to measure a straight line. What is a flat surface? How may it be tested by using a straight-edge; by looking 'along' it, with the eye in the same plane. Why motor headlights reveal faults in the surface of the road.

We have already considered the area of the simple rectangle. It may be revised by solving useful problems on the area of figures which are compounded from rectangles, e.g., large signs in the shape **E T L H** etc., oblong frames or paths round rectangular gardens, where the area of a small rectangle is subtracted from that of a larger. Alternative methods of splitting up the given figure are usually possible, and these may be worked and made to check the results first obtained. Many examples may be found from

fields, gardens, walls, houses, handicraft, drawing, model-making, etc. Other simple rectilinear figures (figures bounded by straight lines) will follow. The properties of a square may be investigated with regard to its diagonals and the numerical relation between side and area will lead to first ideas on 'squaring' and 'square-root'.

The parallelogram is important. It may be demonstrated by knocking out the bottom of a rectangular cardboard box, or using Meccano strips hinged together by bolts. It should be observed that opposite sides and angles are equal, that a diagonal cuts a parallelogram into two identical triangles, and that a rectangle is a particular case of a parallelogram. The area of a parallelogram ($\text{base} \times \text{vertical height}$) is shown by taking a right-angled triangle from one end, inverting it, and fixing it to the other end. A rhombus (diamond) may be demonstrated by joining together four similar Meccano strips. It is a particular case of a parallelogram and may be divided into two isosceles triangles. A trapezium may be regarded as that part of a triangle which remains when it is cut by a line parallel to the base and a smaller (similar) triangle removed.

Other simple figures such as a regular hexagon drawn in a circle (by marking off chords equal to the radius) are worth considering. Rectilinear figures of different shapes which have been found in daily life may be split into simpler rectangles, triangles or squares, and methods devised for determining their areas. Children will be very interested to evolve patterns by the repeated use of simple geometric figures. Many examples of ancient and modern designs may be shown to them: 'stars' and complex polygons, patterns which occur in Nature, in architecture, in domestic art, wallpaper, etc. The planning of gardens and parks will prove interesting. The various 'pattern' numbers (triangular, square, hexagonal and stellate) of the ancients will be useful in helping children with the ordinary proper-

ties of number and notation. 'All things', said Pythagoras, 'are numbers'. Throughout the Senior School there will always arise opportunities for geometrical drawing. From simple figures in early years the better pupils will pass to the making of plans, projections and architectural drawings founded on circles and straight lines. The art and mathematics teachers may collaborate with advantage in many of these tasks. Children should be taught how to keep their simple instruments in good order, the type of pencil to use for a particular task and how to obtain a suitable point, the methods of holding pencil, ruler, set-square and compass, etc. Attention should be paid to the surface on which they are drawing, how it is fixed, the angle at which it is tilted, the position of the light and the general attitude of working. In addition, sufficient instructions should always be given so that the pupil knows where to start, the scale to use, etc. Preliminaries, even as regards scrupulous cleanliness of hands and instruments, count for much if good drawing is required.

A frequent cause of error in early work is a failure to remember instructions and the results of measurements which have been made with the ruler or protractor. These should be written down systematically.

Definitions are important in geometry; and it sometimes happens that the child first realizes, in simple mathematics, the necessity of accurately defining the things which he is going to use. A definition must be as short as possible and be free from ambiguity. On the other hand, a definition does not include all the properties of the thing defined. In geometry one definition follows another, e.g., a rectangle may be *defined* as a parallelogram with one angle a right-angle, but its *properties* include the facts that all its angles are right-angles, its opposite sides are equal and parallel, and its diagonals are equal and bisect one another. In Senior School work it is perhaps not serious if properties occasion-

ally enter definitions. The difficulties attending the framing of sound definitions will readily be appreciated by asking the class to define such common things as table, book, orange, piano, etc. Some of these may prove to be unusually difficult, but the value of such a discussion will aid clear thinking.

The Properties of Triangles

As triangles occur so frequently in daily life, in science and engineering, as the basis of surveying and allied problems, it is well to deal with them thoroughly.

Triangles of various types:

Scalene triangles. A 'general' triangle—all its sides and angles are different.

Isosceles triangle. Two sides are equal. Two angles are equal. It is symmetrical about the perpendicular drawn from the angle made by the two equal sides. It occurs in practice very often, e.g., gables, roofs, faces of pyramids.

Equilateral Triangles. All the sides and angles are equal (and equal to 60°). All equilateral triangles are similar.

The internal angles of a triangle add up to two right-angles. This may be proved:

(1) By experiment.

(2) By producing (extending) each side in the same sense. The exterior angles make a complete revolution (shown by 'walking' round the triangle). The sum of each pair of interior and exterior angles is 2 right-angles. There are three sets of these. Thus the sum of three interior angles = 6 right-angles — 4 right-angles = 2 right-angles.

(3) By producing the base of the triangle to give an exterior angle and from this angular point drawing a line parallel to the opposite sides of the triangle. By a considera-

tion of the angles produced when two parallel lines are cut by a third line it is shown that the exterior angle is equal to the sum of the two interior opposite angles. The third angle completes the two right-angles.

The area of a triangle is equal to half the area of the rectangle with the same base and the same vertical height. The triangle is first shown to be equal to half a parallelogram.

Models may be made of cardboard or wood; paper-cutting and folding should be freely permitted in this early work on demonstrative geometry.

Set-squares may now be considered and used for drawing angles and parallel lines. The properties of the 45° (45° , 90°) and the 60° (30° , 90°) set-squares will appear later when Pythagoras' Theorem (or the 'result' of it) is known. T-squares and try-squares and their uses may be mentioned in passing. Later work with triangles will be taken when the trigonometrical functions are considered.

Instead of the ordinary congruency theorems, which were usually proved by the logically unsound method of superposition, it will serve to consider the data necessary to determine a triangle. Drawings, and experiments with triangles cut from cardboard, will suffice to illustrate the principles.

A triangle is determined if:

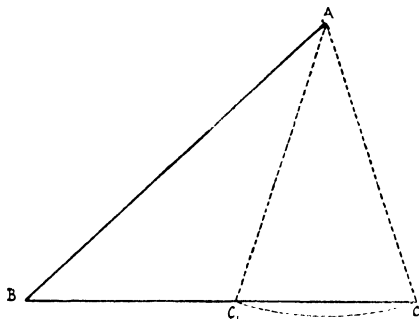
(1) Two sides and the included angle (the angle 'between' the sides) are known.

(2) One side and two angles (similarly placed) are known (if two angles are given, the third may be found by subtracting the sum of the two from 180°).

(3) Three sides are known.

With a right-angled triangle it will suffice to know one side and one other angle. If in case 1, the angle is not 'included', there may be two possible triangles. (The 'ambiguous' case.)

It will be seen from the diagram (or better, by making a 'working' model with strips of wood or Meccano parts) that if the side opposite the angle is smaller than the side which is an arm of the angle (i.e., contains the angle) there

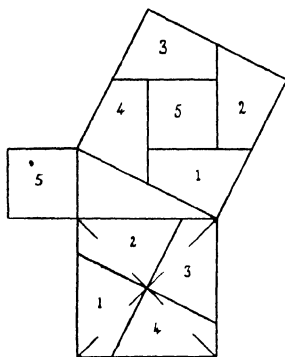


will be two cases. It is a matter of common sense that 3 angles do not determine a triangle for they give us no indication of its size. Equiangular triangles are similar and corresponding sides are proportional. (See Section on proportion.)

We shall meet triangles again when we consider simple trigonometry and surveying.

Pythagoras' Theorem

This most important theorem which finds so many uses in mensuration, trigonometry, pure geometry and mensuration should be approached experimentally by drawing and measuring and through its history. Children are interested in the Chinese method of using triangular and square tiles for making patterns for pavements, and they should lose no opportunity of employing the ancient methods of the Egyptian 'rope-stretchers' for setting out right-angles on the football pitch. The formal proof as given in Euclid is outside the scope of Central School work.



The 'Chinese' Proof. (1105 B.C.)

The square on the larger of the two sides containing the right-angle is divided into four pieces by finding the centre of the square, and through it drawing two lines parallel to two sides respectively of the square on the hypotenuse. The four pieces thus formed and the smaller square are arranged as shown. Children may make models by cutting card or plywood. Many 'area-puzzles' arise from more complex applications of this.

With a top form set of A children it should be possible to give the steps of the general proof. The ordinary text-book proof founded on Euclid may be demonstrated by cutting cardboard, or using coloured 'washes' to fill in the various figures on a single diagram, but it is necessary to know before commencing this proof (1) that the areas of triangles on the same base and between the same two parallels are equal, and (2) the conditions which are necessary for triangles to be identically equal (congruent).

From the Central School point of view the 'result' of Pythagoras' Theorem is the important thing. It must not be forgotten that the square on one of the smaller sides is equal to the square on the hypotenuse minus the square on the

other side. Briefly, Pythagoras' Theorem will find use in:

- (1) Drawing right-angles.
- (2) Finding square-roots by drawing.
- (3) Many applications to trigonometry, surveying and navigation.
- (4) Finding diagonals and other useful distances in figures and solids which contain right-angles.

Children will be interested in drawing right-angled triangles to find square roots. From the formula $n^2 - (n - 1)^2 = 2n - 1$, it follows that the square-root of any odd number may be found by dividing it into two numbers differing by 1, and drawing a right-angled triangle with hypotenuse and one side proportional respectively to these numbers. The other side represents the required square root. Children will be interested to search for sets of numbers such as 3, 4, 5; 13, 12, 5, etc., which will give right-angled triangles.

The properties of circles. How a circle is drawn and defined. The mensuration of the circle and the related plane and solid figures is dealt with separately; but a number of properties of lines and angles in relation to circles may be taken together. By reference to the properties of isosceles triangles, it can be demonstrated that 'the angle at the centre of a circle is double the angle at the circumference'.

From this it follows at once that:

- (a) Angles in the same segment of a circle are equal.
- (b) The angle in a semi-circle is a right-angle. This has important applications in trigonometry and geometrical drawing.
- (c) Pairs of opposite angles of a quadrilateral drawn in a circle (a cyclic quadrilateral) are supplementary (i.e., add up to two right-angles).

Many 'circle properties' may be demonstrated by using the principle of symmetry about a diameter. Tangents may be illustrated practically by considering a taut driving-belt fitted to a wheel. The fact that the 'angle between tangent

and radius is a right-angle' finds application in surveying and navigation, and it gives a clue to the method of drawing and calculating the lengths of tangents by using right-angled triangles.

Even where deductive methods are not used, many teachers still adhere to the order of the geometrical topics as they appear in Euclid, or rather in the modern text-books which have been derived from his work. The sound logical development of the system of Euclid (rather than the faulty logic of some of his proofs) has always appealed to teachers. However, for Central School purposes the 'topic' method of approach may prove to be more satisfactory; and geometry may develop by considering various objects with regard to position, shape and size. The idea of a locus may be demonstrated by paper-folding, by jointed Meccano or wooden strips which can be made to 'draw' the locus under differing conditions, or again by marking out or selecting suitable lines on the surface of the games field, and 'walking' the locus.

Mensuration of Solids

Probably owing to the long influence of Euclid in school geometry, the important practical aspect of the mensuration of solids was frequently overlooked.

The cube may be considered first and built up with layers of unit cubes. There is often some difficulty in proceeding from the measurement of area to that of volume. The idea of a cubic number and cubing will readily follow. The mensuration of solids should proceed with the study of capacities; for not only can a cube of edge 4 inches be built up by using 64-inch cubes, but a hollow cube of tin-plate may be made and its volume found by filling it with water. From the cube we pass to the cuboid or rectangular solid (often called 'oblong box'). By building one from unit layers the formula $L \times B \times H$ for its volume may be demonstrated. As the

mensuration of this solid is so important in our daily life it is well to consider it from every point of view: its volume as found by filling a hollow model with water; the dimensions and total area of its faces; the numbers of its edges and faces; the diagonals of its faces and its internal diagonal, i.e., greatest length of thin rod which may be placed inside it from one bottom corner to the opposite top corner (the latter when the result of Pythagoras' Theorem is known), etc.

Hollow wooden rectangular solids may be treated by making one from six rectangles of wood, or alternatively, by taking a box to pieces. It will be seen, for instance, that top and bottom have 'external' dimensions, the two 'long' sides have internal height and external length, the two ends have internal dimensions and so on. Calculations of the area of wood required may be checked by finding the volume of the wood, by subtracting the internal volume from the external volume, and dividing by its thickness if this is uniform. This, however, is not always practicable.

By sawing up a wooden rectangular block (a cuboid), prisms may be formed. In Central Schools it is only necessary to deal with right prisms, i.e., prisms where the rectangular faces are vertical, when the triangular sides are horizontal. Hollow models may be made from tin-plate. It is seen from experiment, and by building up the prism from triangular layers, that its volume = area of its triangular base $\times$ vertical height. Other solids with parallel sides may also be considered. The prism is very important in practice and is found as part of the form of many buildings.

The trapezoid may be regarded as a cuboid plus one or two prisms. A simple example of a trapezoid occurs in a swimming bath which slopes regularly from the shallow to the deep end. Its volume is equal to the area of a vertical section from one end to the other $\times$ its breadth; and here it is only necessary to multiply length by the average depth to find the area of the section.

A hollow square pyramid may be made from flanged cardboard or tin-plate, and its volume found by filling with sand or water. Various types of pyramidal models with square, rectangular and triangular bases may be used for experiment, and the general rule that $\text{Volume} = \frac{1}{3} \times \text{Base} \times \text{Vertical Height}$ verified. By carefully-made saw-cuts, a prism of wood may be cut into three triangular pyramids, all of which are equal in volume, but this is not always readily seen.

A tetrahedron, octahedron, icosahedron and other more complex solids may be made by paper-cutting, but from the practical point of view it is better to spend the time exploring fully the square and rectangular pyramids, which find considerable application. By Pythagoras' Theorem the slant heights and the length of the edges may be found. The frustum of a pyramid is sometimes met with in practice, and may be regarded as what remains when a smaller pyramid is taken from the top of a larger similar pyramid. In all cases, experiments with paper, cardboard, wooden or tin-plate models should be performed, and the model should be carefully planned and drawn before it is cut out on a plane surface. Practice in obtaining rough estimates of quantities and volumes may be obtained by visiting an 'estate' where building is proceeding, or ships, barges and other craft in sea and river ports. The use and calibration of a 'dip-stick' for finding the capacities of barrels, covered tanks, etc., will also prove interesting.

With A and B, and to a certain extent with C children, geometry may develop by drawing plans and projections which will lead finally to mechanical drawing.

The Mensuration of the Circle, Cylinder, Cone and Sphere

1. The Approach to π .

'And he made a molten sea, ten cubits from the one brim

to the other and a line of thirty cubits did compass it round about.'—First Book of Kings, chap. vii, v. 23.

The nature of the ratio between the circumference of a circle and its diameter has been a tantalizing source of experiment and conjecture for thousands of years. It has entered religion, 'magic', architecture, astronomy, mensuration and science. No single mathematical problem has absorbed so much time as that of endeavouring to evaluate π or trying to find a formula for it. Expressed in the terms of a geometrical construction this may be stated as follows: 'to find the side of a square equal in area to a given circle' and is now known to be an impossible task. This was one of the three great problems of the Greeks, and like the quest for the philosopher's stone in later years, the searches left in their wake many useful discoveries. Some very accurate approximations to π were obtained by Archimedes who used the idea that the area of a circle lies between that of inscribed and circumscribed polygons. By increasing the number of sides the polygons approximated to the circle. Some interesting guesses and approximations to π are given below.

As will be gathered from the quotation referring to the building of Solomon's Temple, the Jews (in common with the Babylonians) thought that π was 3.

$$\pi = \left(\frac{16}{9}\right)^2 = 3.160 \text{ — Egyptians.}$$

$$\pi \text{ lies between } 3\frac{1}{7} \text{ and } 3\frac{1}{4} \text{ — Archimedes}$$

(found by using polygons).

$$\pi = 3.1416 \text{ — Ptolemy}$$

$$\pi = \frac{49}{16} \sqrt{10}, \frac{754}{240} \text{ — Hindoos and Arabs}$$

$$\pi = \frac{355}{113} \text{ (remembered as } 113 \setminus 355) \text{ — Metius (Dutch seven-teenth century)}$$

When it was felt intuitively that no fraction could be found to give an exact value for π , calculators rivalled one another by summing various series of numbers in order to obtain its value correct to considerable numbers of decimal places.

π is not only an irrational but is also a transcendental number, and a mathematical demonstration of its nature which was first given in the nineteenth century was sufficient to show that the circle could never be squared by ruler and compasses.

(*N.B.* Note that geometrical constructions can be found for irrationals, e.g., $\sqrt{2}$, by drawing a right-angled triangle; and it is the transcendental property, i.e., that π can never be expressed as the solution of a rational equation, which precludes the squaring of the circle.)

The importance of π to the mensuration of common objects is evident. Wherever there are circles, crescents (lunes), spheres, etc., π enters. Measurements of angles and trigonometrical functions are often related to π , as elementary trigonometry will show. Therefore it is important that stress should be laid on sound early work. Although the approximation $\frac{22}{7}$ is often used it is apt to prove misleading, and it is usually better to express π as a decimal correct to a given number of places.

Children may commence work by measuring the diameter of a cycle wheel on the rim of which a chalk mark has been made, rolling it over a straight line drawn on the floor, and measuring the length corresponding to a complete revolution or a known number of exact revolutions. This may be varied by:

(1) Using various cylindrical objects such as jam-jars, cocoa tins, pieces of iron piping, etc.; find the diameter by trapping between two rectangular blocks and measuring the distance between them. Then wind strong thread round the cylinder (say) ten times, unwind the thread, measure its length and divide by 10 and then by the diameter. The fact

that the value of π is independent of the diameter of the cylinder will be noted.

(2) A circle is drawn by means of compasses. (Children should be taught to hold the compasses so that a single rotation in the same direction will draw the circle.) The circumference may be estimated by (a) an opisometer (tracing-wheel), (b) by cutting the circle into a number of sectors (each of which is almost an isosceles triangle of small angle), fixing the sectors to paper with paste so that the bases of the triangle are in a 'straight' line. This will then be measured. (c) Or what is practically equivalent to the above—to use dividers opened to exactly half an inch and to mark off this distance along the circumference. This is similar to the polygon method of the Greeks. Many simple practical examples may be founded on the mensuration of the circumference of a circle, e.g., (1) the distance travelled by a revolving wheel and the cyclometer. (2) Edging a circular flower bed. (3) Two driving wheels connected by a belt on a chain. (3) The diameter and circumference of the earth. (4) The distance between the divisions on dials of different diameters, etc.

It should be known thoroughly that the circumference of a circle

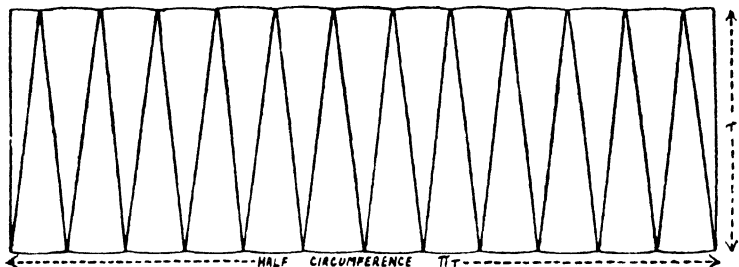
$$= \pi \times \text{diameter}$$

$$= 2\pi \times \text{radius}$$

and that $\pi = 3.142$ (to three decimal places) and a crude approximation, which is easy of manipulation, is $\frac{22}{7}$

The Area of a Circle

(1) A circle of 3 inches radius is divided into a convenient number of sectors (24 is a convenient number), these are cut out and pasted on to a piece of paper as shown. If adhesive paper is used the process is facilitated.



A figure is formed approximating to a rectangle of height = radius of circle, and length = $\frac{1}{2}$ circumference.

$$\begin{aligned}\text{Area of circle} &= \text{Area of rectangle} = \frac{r \times 2\pi r}{2} \\ &= \pi r^2\end{aligned}$$

(2) A circle is drawn on a piece of squared paper, so that its centre is at a convenient point at the intersection of two lines near the centre of the paper, and its radius is equal to an exact number (say 10) of large divisions of the paper. Here r^2 will be 100 sq. units taking each 'large square' on the paper as 1 unit. Now count all the squares enclosed by the circle (or count a quarter of them and multiply by 4) counting as one, all squares of which more than half is within the circle, and neglecting completely all those of which less than half is within the circle. The total number of large squares is then obtained and compared with 100 (representing the radius squared).

(3) A circle is cut from tin-plate, zinc foil or cardboard, and weighed. A square of the same material with sides equal to the radius of the circle is also cut out and weighed. The two weights are compared.

The area of a flat ring, e.g., the path round a circular flower bed or the area of metal in the cross section of a tube, is frequently found in practice. Here the area of the inner circle is subtracted from that of the outer.

If the radii are r_1 and r_2

$$\begin{aligned}\text{Area of the ring} &= \pi r_1^2 - \pi r_2^2 \\ &= \pi (r_1^2 - r_2^2) \\ &= \pi (r_1 - r_2)(r_1 + r_2)\end{aligned}$$

(See Section on Algebra.)

Area of a Sector of a Circle

A sector may be thought of as the 'plan' of a slice of cake, and a segment as the plan of the first piece cut from a round loaf. It will appear from previous work on the determination of the area of a circle by cutting it into equal sectors, that the area of each sector is the fraction of that of the whole circle given by

$$\frac{\text{length of arc of sector}}{\text{circumference of circle}}$$

By experimenting with a rotating arm a child will readily appreciate that the length of arc 'described' by a point on the arm is proportional to the angle through which the arm has turned.

$$\text{Thus, Area of Sector} = \frac{\text{angle of sector}}{360^\circ} \times \text{area of the circle.}$$

The Cylinder

1. The surface of a cylinder.

This may be approached practically by considering a cylindrical tin container, closed at both ends (e.g., as used for 'tinned' pears).

The top and bottom may be removed, the open cylinder cut vertically, opened and flattened to a rectangle. If cardboard or paper models are made and used, the inconvenience of cutting tin-plate will be obviated. We are left with:

$$\begin{aligned}\text{Two circles } 2 \times (\pi r^2) &\text{ in area} \\ \text{A rectangle } 2 \pi r \times h &\text{ in area}\end{aligned}$$

(length of rectangle = circumference of tin, breadth = height of tin = h , r is radius of tin).

$$\begin{aligned}\text{Total} &= 2\pi r^2 + 2\pi rh \\ &= 2\pi r(r + h)\end{aligned}$$

It is always necessary to know if the cylinder is closed or open at one or both ends; or if the area of the curved surface alone is required.

2. The volume of a cylinder.

This may be determined practically by filling cylindrical cans with water. The rule that the volume = area of circular base $\times$ height is practically intuitive (a child realizes that an inch of peppermint rock contains as much as any other inch of the same stick, but that the lower 'half' of an inverted cone contains less than the upper half), but it may be demonstrated by building up a cylinder from circular discs cut from wood 1 inch thick.

$$\text{Volume of a cylinder} = \pi r^2 h$$

Children should be given the opportunity of measuring the diameters of various common objects, and by simple projects discovering means of investigation in difficult cases:

- (1) Use of external and internal callipers.
- (2) Use of sliding callipers or adjustable spanner.
- (3) Use of rectangular blocks, or two set-squares sliding on the edge of a ruler.
- (4) The application of the simple theory of similar triangles to make a tube gauge.
- (5) Indirect method: (a) for finding an external radius. Find the volume of a solid cylinder of known length by displacement of water. Divide volume by length to give area of cross-section (πr^2). Divide by π and find the square root; (b) for finding an internal radius. Fill a hollow cylinder

(a piece of piping) of known length with water. Measure the volume of the water, and proceed as before. For a fine bore mercury is used and weighed. This is more difficult as the weight has to be divided by density to give its volume, and care is necessary with the units employed.

A frequent source of error is a tendency to confuse radius and diameter when working problems. At first it is probably better to convert all diameters to radii by dividing them by 2 before commencing work. Later, the standard formulae may also be thought about in terms of a diameter, which is sometimes the most convenient way.

There is no need to enumerate examples of the cylinder in our daily lives, for it occurs in all types of engineering, in tubes, wires, rods, dowels, pillars, columns, etc. The convenience of a cylinder compared with a rectangular block for many purposes may be discussed. Of all solids having a constant cross-section throughout their length (we may take a sphere as having no 'length') it has the greatest volume for the smallest surface. A pipe with rubber walls would always tend to take a cylindrical shape when it contained a fluid under pressure. Our veins and arteries are cylinders. Flat lead water-pipes tend to become cylindrical when the water freezes. Again, cylinders allow an ease of working on their curved surfaces for they contain no corners; and they can be 'machined' accurately by means of lathes and other rotating machinery.

The Cone

Simple experiments with a hollow tin cone and a cylinder open at one end and of the same base and height as the cone, will demonstrate that the cylinder will hold just three times as much water as the cone. This cone may be made by cutting a sector from a piece of tin-plate, rolling it to make a cone, and soldering together the two radii of the sector.

The Volume of a Cone $= \frac{1}{3}$ volume of a cylinder with same base and height.

$$= \frac{1}{3} \pi r^2 h$$

(h is the perpendicular height)

This may be compared with pyramids and other solids which 'come to a point at the top', and have triangular vertical sections through the apex.

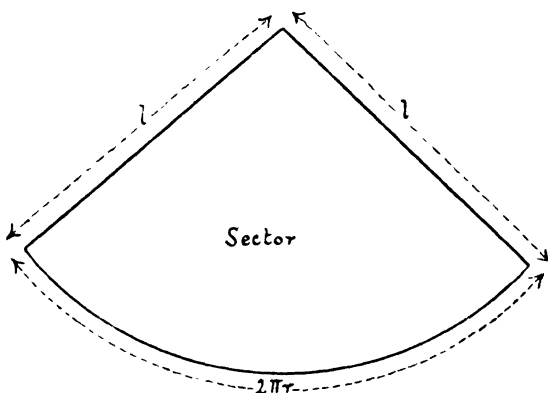
In each case the volume $= \frac{1}{3}$ area of base $\times$ perpendicular height.

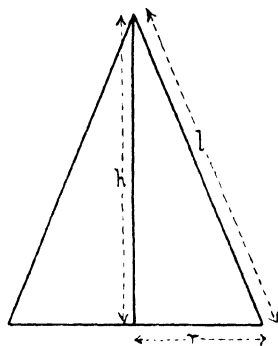
The Area of the curved surface of a Cone

Take a sector of a circle (preferably with an angle greater than a right-angle), bend it until the two radii touch and join them together by means of adhesive paper. It is apparent that:

(a) the slant height of the cone $=$ radius of sector.

(b) the circumference of the base of the cone $=$ length of arc of the sector.





Section of Cone

$$\begin{aligned} \text{Area of curved surface of cone} &= \frac{\text{area of sector}}{\text{length of Arc}} \times \\ &= \frac{\text{circumference of circle of which sector is a part}}{\text{length of Arc}} \times \text{area of circle of which sector is a part} \\ &= \frac{2\pi r}{2\pi l} \times \pi l^2 = \pi r l \end{aligned}$$

If h is vertical height of cone $l = \sqrt{r^2 + h^2}$ (by Pythagoras' Theorem).

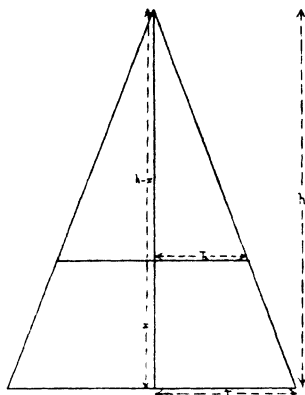
N.B. If total area of a closed cone is required, area of base must be added.

The Frustum of a Cone

This may be demonstrated by modelling a cone in clay or other suitable material, and 'cutting off the top' by a plane parallel to the base. The lower part is called a frustum.

Frustum = Original cone—smaller cone of similar shape cut off.

If we draw a vertical section of a cone we get an isosceles triangle; if a straight line is drawn parallel to the base, the original triangle is cut into a smaller isosceles triangle similar to it and a lower trapezium. The similar triangles provide us with a clue to the working of the problem. (The trapezium is the section of the frustum.)



Given vertical height of frustum = x ; radii of base and top respectively = r_1 and r_2 . Let h = vertical height of cone derived by completing the frustum.

Height of small cone = $h - x$

By using similar triangles

$$\frac{r_1}{r_2} = \frac{h}{h-x} \quad \therefore (h-x)r_1 = hr_2$$

$$h(r_1 - r_2) = xr_1 \quad \therefore h = \frac{xr_1}{(r_1 - r_2)}$$

$$h - x = \frac{xr_2}{(r_1 - r_2)}$$

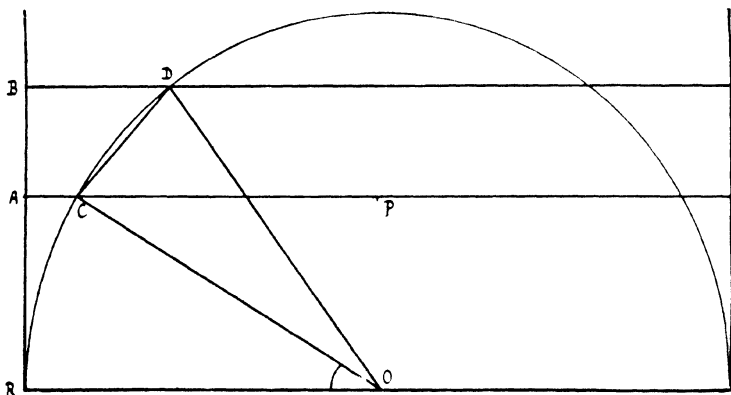
The Volume of frustum = Volume of larger cone - Volume of smaller cone

$$\begin{aligned} &= \frac{1}{3} \pi (r_1^3 - r_2^3) - \frac{1}{3} \pi (r_1^3 - r_2^3) \\ &= \frac{1}{3} \pi x \left\{ \frac{r_1^3 - r_2^3}{r_1 - r_2} \right\} \\ &= \frac{1}{3} \pi x \{ r_1^2 + r_1 r_2 + r_2^2 \} \end{aligned}$$

There are many examples of frusta in daily life, such as flower-pots, vellum lamp-shades, buckets, jugs, etc. The theoretical work should always be supported by experiments or projects, e.g. (1) A vellum lamp-shade has a diameter of a foot at the top and 18 inches at the bottom. Its slant height is 1 foot. Draw a pattern to show how it may be cut out in a single piece. (2) Find the capacity of a bucket by calculation and by direct measurement.

The Sphere

The surface area of a sphere is exactly equal to that of a cylinder which will just enclose it. The proof of this is hardly within the scope of Central School work, though it may be demonstrated by means of similar triangles.



If a corresponding strip is cut, between the same two parallel horizontal planes, from both the sphere and the cylinder into which it just fits, it will be seen that the strip from the sphere is shorter than that from the cylinder; but as it is broader in proportion, the area is exactly the same. By using similar triangles it can be shown that the

radius of the slice of the sphere between A and B is inversely proportional to the length CD (the breadth of the strip). It is essential that the pupil should understand the properties of similar triangles and the 'results' of certain theorems (the angle between tangent and radius is a right-angle).

A knowledge of simple trigonometry will show that the radius (CP) of the strip from the sphere, is proportional to the cosine of the angle

COR, but its width CD (if very narrow) is proportional to $\frac{1}{\cosine}$ of

angle COR (i.e., secant $\hat{C}\hat{O}\hat{R}$) if BA is kept constant in length. The product of CP and CD is thus a constant. This principle finds application in map projections.

Thus, area of Surface of Sphere $= 2\pi r \times 2r = 4\pi r^2$

The Volume of a Sphere

By means of a plasticine model or a piece of Dutch cheese, it can be demonstrated that we may regard the sphere as being composed of a number of small cones each one having its vertex at the centre of the sphere, and a vertical height equal to its radius. The total area of all the tiny bases of these cones will be the area of the surface of the sphere.

Thus:

Volume of the Sphere $=$ Sum of the volumes of all the tiny
cones

$= \frac{1}{3} \times$ their common height $\times$ the sum of the areas of all
their bases.

$= \frac{1}{3} r \times 4\pi r^2$

$= \frac{4}{3} \pi r^3$

Various practical experiments may be made to demonstrate this formula, or alternately, to obtain various data by

assuming it: e.g., the average radius of a number of small spherical pellets may be determined by finding the volume of a thousand of them by measuring the water they displace when they are dropped into a graduated cylinder or burette, dividing to 1000 and applying the formula.

Let v = volume of each pellet

$$\frac{4}{3} \pi r^3 = v$$

$$\therefore r^3 = \frac{3v}{4\pi}$$

$$r = \sqrt[3]{\frac{3v}{4\pi}}$$

Hollow spheres or 'shells' are of frequent occurrence in practice. The volume of metal or other substance in the shell is found by subtracting internal and external volumes. It usually proves to be more convenient to work in terms of internal and external radii. (The thickness of the material = external radius — internal radius.)

The property of the sphere, that for the smallest surface area it encloses the largest volume, has important applications. Add to this the simple fact of surface tension (or the tension of an enclosing elastic membrane) and we see at once the reason for many of the examples of the spherical shape in nature.

Compare the surfaces of a sphere and cube of equal volumes.

It is interesting to compare the volumes of the following solids, by direct immersion of small models in water in a graduated cylinder:

(a) Cone of height equal to diameter of its base. (b) Sphere of radius equal to that of base of cone. (c) Cylinder with same height and base as cone. (d) Cube of side equal to diameter of sphere.

The volumes are in the proportion $1 : 2 : 3 : 4$ (the last is approximate) (i.e., $\frac{2}{3} \pi r^3$; $\frac{4}{3} \pi r^3$; $2 \pi r^3$; $8r^3$).

Orthographic Projection

Orthographic projections are known to the layman as plans and elevations related together by perpendiculars and parallels. The matter may be introduced by considering the necessity of making such drawings before commencing the construction of a building. Actual models should be examined and the plan and elevations drawn. Later, it may be possible to borrow from architects and estate-agents, actual plans and elevations of a building which may be visited and photographed. Many periodicals devoted to architecture, housing, building and carpentry give both photographs and orthographic projections and the pupil should become accustomed to the visualization of the projections by regarding the photograph, and vice versa. In early work it is sometimes helpful to pin the elevations to a vertical board and the plan to a horizontal board in its correct relative position. If a powerful distant arc or 'point-o-lite', placed at the end of a *long* room, is available, projections may be demonstrated by means of shadows. Shadow projections may also be obtained by using light from the sun, but this is not so satisfactory as it introduces an awkward tilting of the object and screen. Although orthographic projections are normally more straightforward than radial projections, and are usually dealt with earlier, it is useful at some stage of the work to discuss both types together and also to refer to the nature of stereoscopic vision.

Using pictures, drawings, photographs and solid objects, it is useful to discuss the difference between perspective and orthographic drawings. A photograph is normally a perspective picture, but a telephotograph taken of a distant façade may approach to an orthographic 'projection'.

Simple orthographic projections will lead to (a) plans and

elevations of furniture, and models made in the school workshop, (b) drawings of farm buildings, poultry-houses, sheds and other structures, which are of particular interest in rural schools, and (c) mechanical-drawing for more advanced pupils. Boys are usually interested in engines and machines; and mechanical drawings of simple examples will involve accurate measurements and calculations, and practice in neat and careful drawing.

Radial Projections

A photograph of a large 'square' isolated house or shed, taken at an angle of about 45° from one side, is a suitable way of introducing the subject. The devices of artists, whose task it is to give the illusion of three dimensions on a plane surface, include the use of perspective. (There are many other methods such as the use of light and shadow, depth of colour, 'focus', etc.) This work may with advantage be connected with corresponding topics in art and the study of light, or again, a few optical illusions depending on the principle of perspective may be demonstrated. (A favourite example shows a long street with its horizontal lines converging towards the horizon. Identical figures of men drawn at various 'distances' along the pavement give the illusion that the nearer figures are smaller.)

Perspective drawing will probably be dealt with in the art lesson but the mathematics teacher will show that all vertical lines have their direction preserved and that parallel horizontal lines converge to meet in a point on the horizon represented by a horizontal line, on which near its centre is the centre-of-vision (a point opposite the eye). Two other horizontal lines are drawn on the paper: (1) the ground line which passes through the nearest point of the object at ground level, and (2) a line through the station point, i.e., a point on the ground vertically below the eye of the observer. From this point lines parallel to the sides of the

object may be drawn to meet the line through the centre-of-vision (usually called the *horizontal* line). The simple theory of the matter may be developed by using similar triangles. When dealing with perspective, 'parallax' methods for finding distances of stars (and terrestrial objects) may also be demonstrated.

A simple model to illustrate radial projections may be made by fixing strong threads to six or seven 'corners' of a large solid cube (or to eight corners of a skeleton cube) so that the threads can be joined at a single point some feet away from the cube and they are all kept taut. If now the threads are so arranged that they pierce a piece of gauze or muslin stretched on a frame (held about half-way between the cube and the point where the threads meet) in such a way that they are all straight and taut, a radial projection of the cube is given by joining the appropriate points where the threads pierce the gauze or muslin. In simple radial projections it is assumed that the object is seen with a single eye or camera-lens.

Architectural scale-drawings of parts of churches and other buildings will prove interesting and useful. Norman, Gothic or Perpendicular windows founded on straight lines and parts of circles when analysed, measured, sketched, photographed and drawn accurately will reveal the beauty of craftsmanship and link the work to history and architecture. Children are interested to find the simple numerical relations which often exist between the chief measurements of a building or a part of it. Scale drawings of classical structures, and buildings in other more recent styles, will reveal 'harmonic' relationships on measurement. The eye is often quick to detect small departures from these related distances, and we say the building is ugly, disproportionate, top-heavy and so on. The word 'harmonic' is used by analogy with music, where simple ratios between the frequencies of notes yield concords which sound pleasant to

the ear, e.g., a fifth (Doh to Soh, or C to G) has a ratio 2 : 3; a third (Doh to Me or C to E) a ratio 4 : 5; a fourth (Doh to Fah, C to F) a ratio 3 : 4 (speaking ideally, and not in terms of the tempered keyboard scale). Of course the matter is psychological as well as mathematical, and recent investigations have by no means exhausted its potentialities. Similar principles apply widely in Nature both animate and inanimate, as well as in man-made objects. An efficient animal-structure or machine may be beautiful because it has been designed to fulfil certain mechanical principles which may be expressed in formulae. The whole subject may be developed in many fascinating and often difficult ways along lines which combine æsthetic, psychological, mechanical, and mathematical considerations. In this connection teachers will be interested to read *Engines of the Human Body* (Keith); *Growth and Form* (D'Arcy Thompson); *The Psychology of Beauty* (Valentine).

CHAPTER X

TRIGONOMETRY, SURVEYING, NAVIGATION, MAP-PROJECTIONS

1. *Trigonometry*

Formerly, trigonometry was studied as a separate topic with perhaps occasional references to the geometry of triangles and circles and simple algebraic transformations. Trigonometry 'put number back into geometry', and besides its great practical utility which will increase its appeal to the ordinary pupil, it offers useful connecting links with geometry, algebra and graph drawing. In other words, it is a great help when we try to show the essential unity of mathematics. The very simplest ideas of trigonometrical ratio prove to be of enormous utility in the sciences both pure and applied. The boy who has already undertaken some 'Boy-Scout' surveying will discover that trigonometrical tangents are a great help to him; and the student studying light will find that the law of Willebrod Snell (the 'Law of Refraction') expressed in any other way than 'the sine of the angle of incidence divided by the sine of the

angle of refraction is a constant, $\frac{\sin i}{\sin r} = \mu$, is exceedingly

tedious and clumsy; besides which, if the student is going to investigate the matter any further the sine law may give him a clue to form a theory as to what happens when a ray of light is refracted.

The term Sine, and the new way of regarding the terms Tangent and Secant (which have already been used in the geometry of the circle) need some explanation. Early work in trigonometry should aim at showing:

(a) That it grows naturally out of previous work in algebra and geometry, including surveying and drawing.

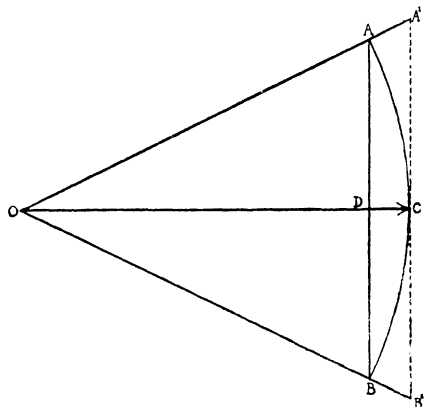
(b) That it is a tool of very wide application, simplifying or even making possible the work of surveyors, engineers, navigators (and many 'pure' mathematicians).

(c) That the terms used can be justified historically and logically.

In formal trigonometry the approach to the subject has usually been through a consideration of the nature and measurement of angles, right-angled triangles, their properties and the simple trigonometrical functions, the inter-relations of sine, cosine, tangent, etc. Although it may be necessary to return to consider radian measure and related topics, in Senior Schools it will suffice to start trigonometry through its name—the measurement of triangles. This may be done both practically and historically. Already, the children will probably have found the heights of trees, the school flag-pole and the church spire by measuring the length of a shadow, and at the same time the length of the shadow of a vertical pole (a Scout stave of 6 feet) and comparing them by drawing. Or alternatively, they will find the angle of elevation of the top of the flag-pole by means of a clinometer, measure the distance from the observer to the bottom of the pole, and draw a scale diagram (care being taken to add the distance from the ground to the observer's eye). Children will be interested in some quick method which will enable them to omit the drawing of the right-angled triangle to scale and the measurement of its vertical line.

Although the simple ideas of trigonometry were known at a much earlier date, particularly by the Arabs and Greeks, the word Sine (Latin Sinus) does not appear until the end of the twelfth century, when it was introduced by Gerard of Cremona. The diagram shows how the early trigonometrical functions were not represented by ratios, but rather by lines.

As they were all referred to the length of the radius of the circle (i.e., the radius vector), if we were to convert them to ratios these would, at least, have the one advantage that the length of this line would be the common denominator.



In the diagram $OA = OC = OB$ (the radius) OA^1 (a line *cutting* the circle) was called the Secant¹, CA^1 (a line *touching* the circle) was called the Tangent.¹

By the simple theory of Similar Triangles it is seen that

$$\frac{OA^1}{OC} = \frac{OA}{OD} = \text{Secant (as used at present) and}$$

$$\frac{CA^1}{OC} = \frac{AD}{OD} = \text{Tangent (as used at present)}$$

The early writers were not slow to see that a diagram such as the above, bore a resemblance to a bow (ACB) with an arrow (OC) stretching the string AOB, before it was released. Indeed, for a long time the line OC was called

¹ The term 'sine' preceded the use of 'tangent' and 'secant' in trigonometry by many years.

Sagitta (arrow). The line AD was called the Sinus (bosom or breast line), hence our word Sine.

In our modern style

$$\frac{AD}{OC} = \frac{AD}{OA} = \text{Sine (as used at present)}$$

The 'co' prefixes are obvious if one considers such relationships as 'the cosine of an angle is (numerically) equal to the sine of its complement', which can readily be proved by considering the ratios between the sides of a right-angled triangle taken in pairs.

When the desirability of using the ratios of pairs of sides of a right-angled triangle has been considered by the children through work in measuring heights and distances and 'Boy-Scout' surveying, they may begin with the relations between the sides of a right-angled triangle. Simple experiments with similar triangles will show that if triangles are equiangular they also are similar (they have the same shape). As the three angles of a triangle add up to 180° , a knowledge of two of them will fix the shape of any triangle. Thus, if a triangle is right-angled the shape of the triangle is determined, if another angle is known. It follows then, that for every value between 0° and 90° which we may give one of the remaining angles of a right-angled triangle, there will be a perfectly definite shape, and a definite set of ratios between pairs of sides.

$$\begin{array}{lcl} \text{Sine of angle} & = & \frac{\text{perpendicular}}{\text{hypotenuse}} \left(\frac{P}{H} \right) \\ (\text{Sin.}) & & \end{array}$$

$$\begin{array}{lcl} \text{Cosine of angle} & = & \frac{\text{base}}{\text{hypotenuse}} \left(\frac{B}{H} \right) \\ (\text{Cos.}) & & \end{array}$$

$$\begin{array}{lcl} \text{Tangent of angle} & = & \frac{\text{perpendicular}}{\text{base}} \left(\frac{P}{B} \right) \\ (\text{Tan.}) & & \end{array}$$

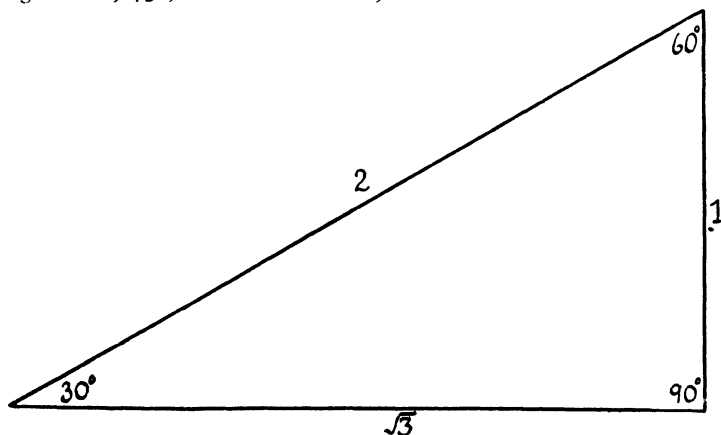
The reciprocals of these: the cosecant, secant and co-tangent respectively, need not be learnt at this stage.

It should be stressed that if we know the value of any one of these ratios all the others may be determined, either by drawing a right-angled triangle and applying Pythagoras' Theorem, or by a simple manipulation of the formulae by which the trigonometrical functions are connected, viz.—

(1) (A is an angle) $\text{tangent } A = \frac{\text{sine } A}{\text{cosine } A}$ which follows by merely dividing the ratios as set out above.

(2) $\text{Sin}^2 A + \text{cos}^2 A = 1$ which follows from Pythagoras' Theorem.

Tables of sines, cosines and tangents may be constructed for angles of 30° , 60° , 45° , and later 0° and 90° .



The 30° , 60° , 90° triangle may be regarded as half an equilateral triangle. (Examine a set-square of this form.) By Pythagoras' Theorem the sides are in the ratio 1. $\sqrt{3}$. 2.

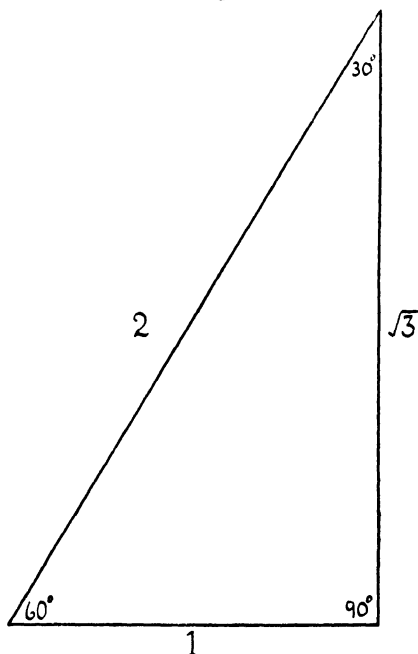
Thus $\sin. 30^\circ = \frac{1}{2}$

$$\cos. 30^\circ = \frac{\sqrt{3}}{2}$$

$$\tan. 30^\circ = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

In the same way with an angle of 60° :

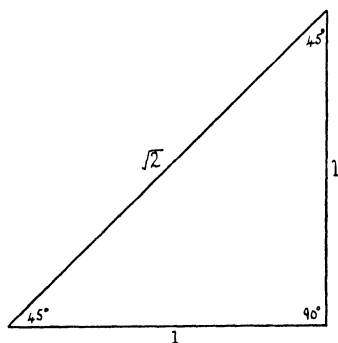
$$\begin{array}{l} \sin. 60^\circ = \frac{\sqrt{3}}{2} \\ \cos. 60^\circ = \frac{1}{2} \\ \tan. 60^\circ = \sqrt{3} \end{array} \left. \begin{array}{l} \text{Notice here the complemen-} \\ \text{tary relationship with the} \\ \text{trigonometrical functions of} \\ 30^\circ. \end{array} \right\}$$



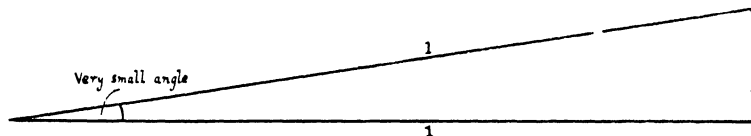
In a $45^\circ, 45^\circ, 90^\circ$ triangle two sides are equal and the other side may be found by Pythagoras' Theorem.

The sides are in the ratio $1, \sqrt{2}, 1$.

$$\begin{array}{l} \sin. 45^\circ = \frac{1}{\sqrt{2}} \\ \cos. 45^\circ = \frac{1}{\sqrt{2}} \\ \tan. 45^\circ = 1 \end{array}$$



To obtain the trigonometrical functions of 'an angle of 0° ' we must take a right-angled triangle with a very small angle and imagine what happens when this angle approaches 0° .



The base and the hypotenuse may be considered equal and the perpendicular very small, and it will tend to disappear as the angle approaches 0 .

$$\sin. 0^\circ = 0$$

$$\cos. 0^\circ = 1$$

$$\tan. 0^\circ = 0$$

A similar method may be used to determine the trigonometrical functions of an angle of 90° .

By using generating circles the graphs of sines, cosines and tangents may be plotted. This is worth doing from several points of view:

1. It shows that sines, cosines, etc., are continuous functions and 'repeat' each 360° . It gives a quick way of estimating the values of these functions for various angles; and the memory of the shapes of the graphs will give a clue to the calculation of the functions of second, third and fourth quadrant angles. The pupil will at once grasp that in

calculating the functions of angles greater than 90° , they must be regarded as angles of $180^\circ +$, $180^\circ -$, or $360^\circ -$. Such relations as: 'the sine of an angle is equal to the sine of its supplement' can be seen from the graph. A mnemonic to show which functions (and their reciprocals) are positive in any quadrant, is given below:

$\begin{array}{c} 2 \\ (180^\circ -) \text{ Sine} \\ (180^\circ +) \text{ Tan} \\ 3 \end{array}$	$\begin{array}{c} 1 \\ \text{All} \\ \text{Cos } (360^\circ -) \\ 4 \end{array}$
--	--



2. The sine-curve is of the utmost importance for later work in mathematics and the applied sciences. All periodic functions can be regarded as being formed by adding sine-curves, the frequencies of which are in the ratios of the natural numbers. Tides and other astronomical occurrences are calculated by compounding sine-curves; and musical tones can be split up into harmonics each of which is represented by such a curve. (There is an ingenious instrument which will generate any musical tone-quality by adding tiny electric currents in the form of sine-waves.) The electrical engineer dealing with alternating-current finds that smaller sine-curves appear; and are added to the main sine-curve, representing the alternation of the voltage. The wireless enthusiast who uses the terms 'second-harmonic distortion' and 'percentage modulation' cannot really understand what he is talking about, unless he knows how curves may be combined by adding the ordinates (in the first place with a simple ratio between frequencies, and in the second usually no simple ratio). The curve will demonstrate a principle which may require quite complex mathematical elucidation. Many Central School boys continue with technical work and various forms of engineering and then some knowledge of simple-harmonic-motion and wave-propagation is useful.

3. When radian measure has been grasped, the abscissae of the graph will be marked off as $\frac{\pi}{2}$, π , $\frac{3\pi}{2}$, 2π , instead of 90° , 180° , 270° , 360° .

If the scales along both axes are the same (the angles being measured

in radians) it will easily be seen that the graph goes through the origin at an angle of 45° to the x axis, which is equivalent to saying that for a small angle θ (in radians): $\text{Sin. } \theta = \theta$.

This is the 'small angle' so often used in physics when dealing with such things as the theory of the pendulum. It is interesting to plot the errors made by using this formula $\text{Sin } \theta = \theta$ for different angles (θ being expressed in radians).

Radian measure offers no difficulties. A radian is the angle of a sector of a circle with an arc equal to its radius. A chord equal to a radius would make an equilateral triangle and therefore subtend an angle of 60° at the centre. Hence a radian is slightly less than 60° .

$$2\pi \text{ radians} = 360^\circ$$

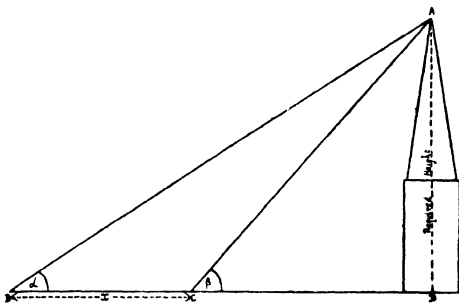
$$\therefore 1 \text{ radian} = \frac{360^\circ}{2\pi} = \frac{180^\circ}{\pi} = 57.3^\circ \text{ (approx.)}$$

4. If cosine and sine-curves, which are identical in everything except position, are drawn on the same graph their complementary relationship may be shown. In later years when tangents to curves and differential coefficients are thought about, the differential relationship between sine and cosine may be demonstrated.

5. Lastly (and most simple), tangent and sine-curves (particularly the first) may be drawn by the children and used instead of tables for solving problems which arise when surveying is undertaken.

Application of Trigonometrical Functions to Simple Problems

1. Finding the height of a distant spire, the base of which cannot be reached:



Given distance x and angles α and β .

Find height of steeple AD (h).

As the only distances to be considered are horizontal and vertical, sines and cosines and their reciprocals (all of which contain the hypotenuse) are excluded. Tangents or cotangents of the angles will be required. Where there is a height AD common to both triangles it is often more convenient to use this as the denominator of the ratio, i.e., use cotangents.

$$\cot. \alpha = \frac{BD}{h}$$

$$\cot. \beta = \frac{CD}{h}$$

$$\therefore \cot. \alpha - \cot. \beta = \frac{BD}{h} - \frac{CD}{h} = \frac{x}{h}$$

$$\therefore h = \frac{x}{\cot. \alpha - \cot. \beta}$$

or using tangents:

$$BD = x + CD = \frac{h}{\tan \alpha}$$

$$CD = \frac{h}{\tan \beta}$$

$$\text{Subtracting } \therefore x = h \left(\frac{1}{\tan \alpha} - \frac{1}{\tan \beta} \right)$$

$$\therefore h = \frac{x}{\left(\frac{1}{\tan \alpha} - \frac{1}{\tan \beta} \right)} = \frac{\tan \alpha \tan \beta x}{\tan \beta - \tan \alpha}$$

(If the angles were taken with a clinometer or similar apparatus, do not omit to add the height of the observer.)

Variants of this problem may be made by considering the 'angles of depression' at different times of a ship sailing to

or from the observer. If the speed and time are known the height of the cliff may be calculated. If the height of cliff and interval of time are known, the speed of the ship may be calculated and so on.

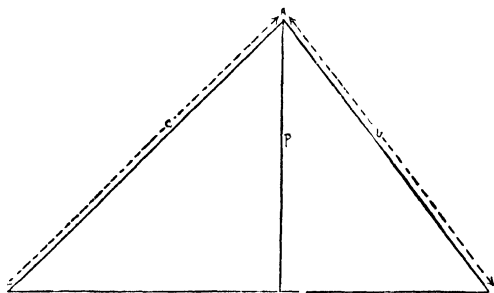
The science master will be able to suggest some further applications of the simple trigonometrical functions in mechanics (equilibrium, and the resolution of forces); light (law of refraction); sound (waves and tone quality); electricity and magnetism (resolution of forces, galvanometers, etc.). In surveying, projections of various kinds, navigation, and astronomy we find many more uses for these functions.

No trigonometry of a formal type will normally be undertaken in the Modern School but the simple trigonometrical properties of triangles are so important in their applications and present such little difficulty, that it may be possible to consider them.

1. Sine Rule.

$$\frac{\text{Sine } A}{a} = \frac{\text{Sine } B}{b} = \frac{\text{Sine } C}{c}$$

Consider first the acute angled triangle: ABC . Notice the conventional way of recording the length of the sides, side a opposite angle A , side b opposite angle B , etc. Draw the perpendicular AD of length p .



$$\text{Sine } B = \frac{p}{c} \quad \text{Sine } C = \frac{p}{b}$$

$$\text{Sine } B \times c = \text{Sine } C \times b$$

$$\frac{\text{Sine } B}{b} = \frac{\text{Sine } C}{c}$$

As the triangle is a Scalene triangle we have not made any assumptions with regard to its sides, thus without further proof:

$$\frac{\text{Sine } C}{c} = \frac{\text{Sine } A}{a}$$

$$\text{or} \quad \frac{\text{Sine } A}{a} = \frac{\text{Sine } B}{b} = \frac{\text{Sine } C}{c}$$

Simple algebra will serve to produce new arrangements of this simple formula. It will be applied to the finding of distances and angles when surveying by means of triangulation, but to be able to use this formula, we must know one angle of the triangle and the length of one side opposite to it (and one other measurement).

In the case of a triangle containing an obtuse angle the matter is not rendered much more difficult for the sine of an angle is also equal to the sine of its supplement, and we consider the triangle containing the acute supplementary angle. The sine-rule applied to the 'triangle of forces' gives Lami's Theorem (when 3 forces acting at a point are in equilibrium each force is proportional to the sine of the angle between the other two).

The Cosine Rule

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

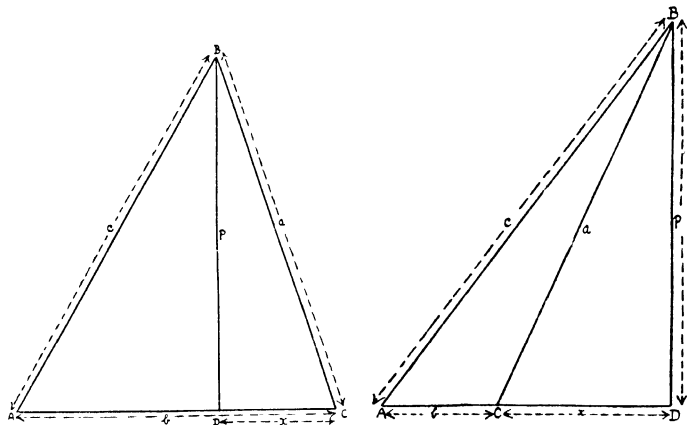
$$c^2 = a^2 + b^2 - 2ab \cos C$$

which when rearranged yield

$$\cos. A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos. B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos. C = \frac{a^2 + b^2 - c^2}{2ab}$$



There are several ways of approaching the cosine rule. If a triangle is explored algebraically by drawing a perpendicular to form two right-angled triangles and Pythagoras' Theorem is applied we obtain the proof in the easiest way:

Using the lengths given in the diagram

$$\begin{aligned} \text{From triangle ABD } c^2 &= p^2 + (b - x)^2 \\ &= p^2 + b^2 - 2bx + x^2 \quad \dots (1) \end{aligned}$$

$$\text{From triangle BDC } a^2 = p^2 + x^2 \quad \dots (2)$$

$$\therefore c^2 = a^2 + b^2 - 2bx$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\left(\cos C = \frac{x}{a} \quad \therefore x = a \cos C \right)$$

x is called the projection of a on AC . The length of such projection is found by multiplying the original length ' a ' by the cosine of the angle between the original line and the line on to which it is projected.)

In the case of an obtuse-angled triangle the cosine-rule still applies, but as will be seen from a diagram, we get (following the same steps as before)

$$c^2 = a^2 + b^2 + 2ab \cos \hat{A}CD$$

but $\hat{BCD}$ is the supplement of C

$$\therefore \cos \hat{BCD} = -\cos C$$

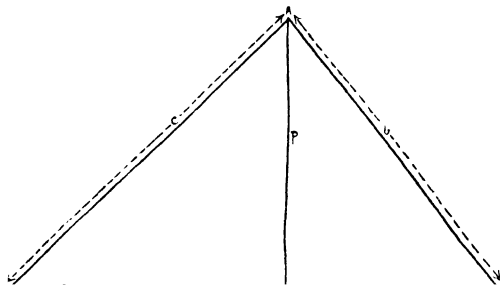
Thus again

$$c^2 = a^2 + b^2 - 2ab \cos C,$$

and similarly for a^2 and b^2 . The cosine rule is useful in finding the angles of a triangle when only the sides are given, or when two sides and the angle between them (the 'included-angle') are known.

Notice that the cosine rule contains Pythagoras' Theorem as a particular case.

The Area of a Triangle



$$\text{The area of a triangle} = \frac{\text{base} \times \text{perpendicular height}}{2}$$

$$\frac{ap}{2}$$

Now $p = c \sin B$

$$\text{Area of a triangle} = \frac{ac \sin B}{2}$$

and similarly (by thinking of each side in turn as base)

$$\text{Area of a triangle} = \frac{ac \sin B}{2} = \frac{ab \sin C}{2} = \frac{bc \sin A}{2}$$

or in words, multiply together the lengths of two sides and the sine of the angle between them and divide by 2.

These theorems are very useful because surveying relies very largely on the use of triangles and their properties.

Surveying

Surveying in schools may commence with considerations of making maps, plans and sketches. The requirements for Boy Scout badges will yield useful problems; and the geography, art and crafts masters will be able to supply helpful suggestions. The properties of similar figures should be revised, but such trigonometry as is necessary may be evolved as the subject develops. Simple work in scale-drawing will have been undertaken already when making plans and elevations of models, reading maps, etc.

The simple arithmetic involved in using scales is not a matter of any difficulty, although calculations of area and volume from scale drawings are frequently overlooked (see the section on proportion). The choice of suitable scales for various purposes of plan- or map-drawing should be discussed from a practical point of view, e.g., the school playing-field is $\frac{1}{4}$ -mile long, and not as broad as long; we wish to draw a plan not more than two feet in length, what will be a convenient scale? What is the scale of the large school globe? Stretch a piece of string from one port to another to represent the path of a ship or aeroplane travelling along a great circle. Find the actual distance by reference to the scale. Find the distance travelled by a day's walk or car

ride in the country by taking an ordnance-survey map, marking out the route with thread, measuring this, and converting by means of the scale.

Taking a large-scale map, trace the outline of a lake, a park or the boundaries of a city on squared paper (with inch squares divided into tenths), estimate the number of squares by counting, and find the area in square inches on the map. Find the real area from the scale, remembering that here we are dealing with square measure and if the scale is $1 : x$ the real area is x^2 times that of the area of its representation on the map.

Many of the simple surveying instruments used in schools may be constructed in the workshop.

1. Measurement of length.

Use of 100-link chain and tape; accuracy in pacing (boys should experiment for themselves); estimation of length by time taken at various speeds (e.g., scout's pace); estimation of distance or length by inspection; scouts will know that distances seen from a height and over water are sometimes deceptive; use of 'parallax' in estimating distances (this may be demonstrated in the class-room, from observations taken from the windows of railway carriages; and the 'parallax' of fixed stars due to the motion of the earth may be mentioned). The history of chains, rods and poles may also be mentioned.

Chain-Surveying

It is recommended that surveying should, if possible, be undertaken in the country away from the school. It may form part of the work of school and scout journeys or camps. A field-book for recording measurements will be necessary and the entries should start at the bottom of each page and work upwards. Two vertical lines about $\frac{1}{2}$ inch apart should be ruled throughout the length of the page leaving an equal space on both sides. In the column between

equal in all respects. Other simple methods should be sought.

The history of mathematics yields many interesting examples of the use of triangles in surveying land. Thales, who lived about 600 B.C., used methods very similar to the above.

An example of finding heights and distances of a more difficult type has already been given, but some simple introductory methods are worth performing.

Finding Heights of 'Accessible' Objects

1. A 45° set-square with a plumb-line attached to one of the angular points, so that one of the shorter sides may be held vertical, may be used for finding the height of an object the base of which is approachable. The top of the object is sighted so that it is in line with the long side of the set-square. The observer walks forward until one of the shorter sides of the set-square is vertical. The distance of the observer from the base of the object plus the distance between the ground and the observer's eye is then equal to the height of the object.

2. The length of the shadow of a 6-feet scout staff is compared with that of the object *measured at the same time*.

From the properties of similar triangles:

$$\frac{\text{Height of Object}}{\text{Height of Staff}} = \frac{\text{Length of Shadow of Object}}{\text{Length of Shadow of Staff}}$$

$$\therefore \text{Height of Object} = \frac{\text{Height of Staff} \times \text{Length of Shadow of Object}}{\text{Length of Shadow of Staff}}$$

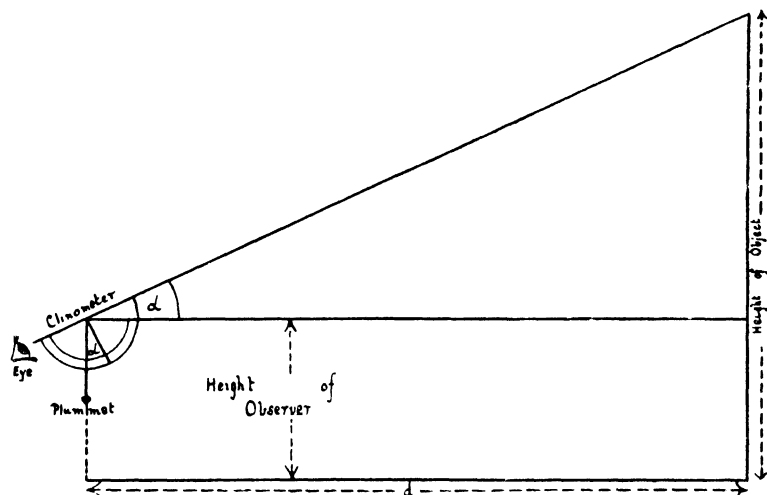
The result may also be found by drawing to scale.

3. A clinometer may be improvised by taking a large projector and fixing to the centre (of the circle of which it is a part) a plumb-line. It is better, however, to construct the instrument properly by using a half circle of wood graduated with a scale reading from 0° to 90° on each side of the 'horizontal point'. The plummet may be constructed by using black thread and a small weight. The angle of elevation α , the horizontal distance of the observer from the object and the height of the observer's eye above the ground must be found. The height of

the object is found by making a scale drawing, or by using the tangent of the angle α .

Height of object = Distance (d) $\times$ $\tan \alpha$ + height of observer (h).

4. In another method the eye (as near as possible to the ground level) is 'put into line' with the top of a vertical staff of known height and the top of the object whose height is to be measured. The distances between the eye and the base of the staff and the eye and the base of the tower are measured. A scale-drawing is made and the simple proportions of similar triangles used in the calculation.

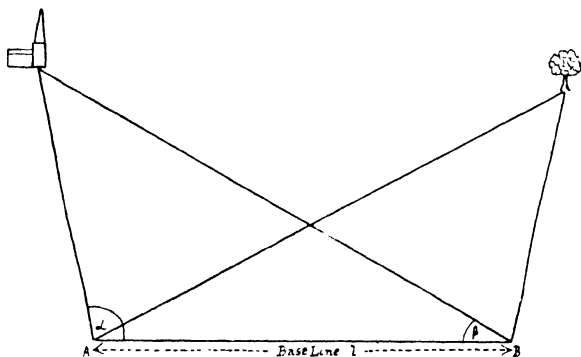


Triangulation using a Base of known Length

In chain-surveying using triangles, three sides have to be measured, but fixing a base-line of known length and taking bearings on distant objects from both ends of the base-line not only simplifies the work considerably, but makes easy surveys which would otherwise be difficult or impossible. It will be remembered that a triangle is determined if one side and two angles are known.

Surveying from a base-line has great importance and pupils will be interested to learn something of the history of

the matter. In England, a base-line was marked out on Hounslow Heath and the survey proceeded from this by triangles throughout the British Isles. 'A base of verification' nearly seven miles in length was measured on Salisbury Plain. The degree of accuracy required for such surveys, both in England and in the Colonies, gave a great incentive to physical investigations on gravity, pendulums, the expansion of metals, etc. The range-finder with a base of known length and a mirror or telescope at each end, depends on a similar principle of triangulation, and may be mentioned.



The base-line should be at least 2 chains long, and measured out accurately on horizontal ground which commands a view of prominent distant objects, such as a chimney, a church, an isolated tree, the top of a barn, etc.

The bearing of the base-line may be found by observations on the sun or by using a compass (in the latter case making the requisite adjustment, taken from *Whitaker's Almanack*).

For measuring the angles of the bearings of the distant objects at each end of the base-line we may use a circular scale graduated in degrees (fixed to a horizontal drawing-board placed on a tripod), or a theodolite; or again, a plane-table method may be used, and the plan of the base-line and

the distant objects drawn as the work proceeds. When using a tripod, it should be firmly fixed on the ground so that its centre is directly over the end of the base-line, and the instrument should be levelled. In taking angular measurements, care is taken not to touch the instrument except when rotating the telescope or sighting-tube. If a round of angles is taken, a check on the working may be obtained by adding the angles together to see whether their total is 360° . It is possible to make a serviceable theodolite in the workshop, but the survey will be more realistic if a good instrument capable of both horizontal and vertical readings can be borrowed.

The plane-table offers the best way of simple and quite accurate plan drawing by triangulation from a measured base. The apparatus consists of a tripod on which is fixed a horizontal board, a good magnetic-compass, a sighting-ruler and a spirit-level. The plane-table is used as follows: It is set up and levelled (two positions of the spirit-level at right-angles to each other will suffice) over the position A at the end of the base-line. A piece of drawing-paper is carefully pinned to the board. The direction of the magnetic north is marked on the paper, a point near the centre of it is chosen to mark point A at the end of the base-line. With this point touching one edge of the sight it is turned round until it is in line with the point B at the other end of the base-line, now marked by a vertical ranging-rod. A line is drawn on a suitable scale to represent the base-line. The ruler is now sighted to a distant object the line of which is immediately drawn from point A. This is repeated with other objects, until series of rays each bearing the name of the distant object have been drawn from point A. The table is then removed from A, the position of which is marked by a ranging rod, and taken to point B. Here a similar set of operations is performed.

This may be varied by simple magnetic-traverses by

means of a prismatic compass. Rough work may be performed by using an ordinary compass and pacing the required distances. One of the chief tasks of the surveyor is levelling. This is normally performed by a telescopic level of known height above the ground, which is sighted upon a graduated-staff held in a vertical position at a measured distance away. Gentle slopes may be broken into a series of triangles, and the vertical rise for each distance travelled along the slope noted in the field book. Steep gradients may be determined by using the clinometer or theodolite. The levels may be referred to the ordnance bench mark ($\overline{\Lambda}$), of which the positions and height above sea-level (the high-water mark at Liverpool) are marked in the 6-inch ordnance survey maps. This will lead to a consideration of gradients and contours. In regard to the former it must be clearly stated whether a gradient of 1 in x means a rise of 1 foot vertically for every x feet along the slope or along a horizontal line. With small angles of slope there is little difference.

Contours may be demonstrated by making models of the hill in clay and cutting through in horizontal planes. Although it may be more properly undertaken in geography lessons, some discussions should be held on the history of map-making, modern methods of surveying, the ordnance survey maps and their conventional signs.

With younger senior school children 'surveying' may take the form of hunting for buried treasure and boy-scout games of a similar nature. Many adventure stories which are popular with young people contain references to clues which give distances and directions. The chart found at the bottom of the seaman's chest still carries a thrill with the secret it holds. Children will be interested in the types of stories which have been evolved since *The Gold Bug* was written by Poe. The islands mentioned in *Treasure Island*, *Robinson Crusoe*, *Swiss Family Robinson*, *Coral Island*, *Gulliver's Travels* and in many other stories which have

appeared more recently may be drawn; or children may imagine that an island on the Pacific Ocean contains treasure, they may find a suitable place by examining a globe, and then write a story giving the clues.

Many of the historical attempts at surveying and map-making are of great interest and children will like to see models of the simple instruments available to the ancients. Eratosthenes was able to measure the circumference of the earth by finding that when the sun was directly overhead at Syene (Assouan) on the Tropic of Cancer, it was $7\frac{1}{2}$ degrees from the zenith at Alexandria, 500 miles away. Thus he argued that the circumference of the earth is $\frac{360}{7\frac{1}{2}} \times 500$

miles or 25,000 miles, which gives a radius of approximately 4,000 miles. Hipparchus, who made a table of numbers which we should now call 'sines', estimated the distance of the moon by using angles obtained by observations from two places and applying the properties of a right-angled triangle; and Aristarchus (310 B.C. to 230 B.C.), who anticipated to some extent the ideas of Copernicus and Galileo on the heliocentric theory, was able to make a rough comparison of the distances of the moon and sun from the earth.

Navigation

This subject presents a very wide field, particularly since air navigation has yielded its own problems, and wireless signals have assisted the navigator both in the air and on the sea. The subject is full of historical interest in connection with our maritime achievements, and it may be developed from work on time, map-projections, and graph-drawing.

Longitude and Time

The earth makes a complete revolution (or turns through 360°) in 24 hours; thus, in an hour it turns through $\frac{360^\circ}{24}$

or 15° . 15° of longitude are therefore equivalent to an hour's difference of time. Now, the sun rises in the east and sets in the west and the earth is revolving 'from west to east'. As we go eastwards from a certain point on the earth's surface we find that the new time is ahead of ours and we have to put our watches on, perhaps to the extent of only an hour, or half an hour, if we are visiting central or western Europe. If we sail west towards America we find that we have to put our watches back, and when we reach the east coast of America time is five or six hours slow compared with English time.

If we are voyaging round the world in this direction, i.e., England to New York, New York to the Pacific coast, across the Pacific Ocean and so on, we are elongating every day and thus at some convenient place before we return we have to add one to the reckoning of the number of days (this is done by 'jumping' from Monday to Wednesday for example), and vice versa. When time can be determined by observations on the sun or stars at any particular place, and this time compared with Greenwich time obtained from an accurate clock or by wireless signals, at once we have a means of determining the longitude of that place, for every hour of difference represents 15° . It can easily be understood why, during the past centuries, so much skill has been applied to the task of constructing accurate clocks, called chronometers, for the purpose of finding longitude at sea. (It must be understood that there is nothing absolute about time, and that the position of Greenwich, as the place through which passes the line of zero longitude, is quite arbitrary.)

Latitude

Take a large globe suitably mounted and draw a circle to go through its poles and Greenwich. This is a meridian, and like all circles of longitude it is a great-circle (i.e., its

diameter is equal to that of the earth). Now draw the equator and consider the arc of 90° on the circle of longitude between the equator and the pole. Each degree of latitude

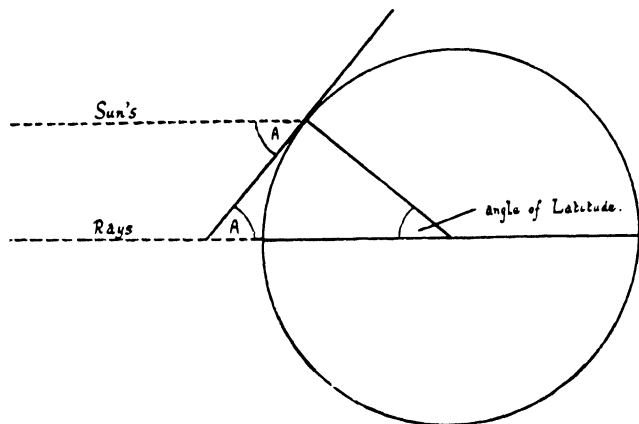
$$= \frac{\text{length of earth quadrant}}{90} \quad \text{or}$$

$$\frac{\text{length of polar circumference}}{360} = 69 \text{ miles.}$$

(*N.B.* As the earth is an oblate spheroid the lengths of polar and equatorial circumferences are not quite equal, but along the equator a degree of longitude is practically 69 miles.) A minute of latitude (or longitude at the equator) = $\frac{69}{60}$ miles. This is called a sea-mile, and it is at once evident that it is a convenient unit for navigators. A knot is a speed of one sea-mile an hour; it derives its name from the knots in the rope which was 'paid out' when the 'log' was thrown overboard. If the knots were placed at equal distances along the rope, the length of rope which left the ship in a given time, measured in terms of these knots, would be proportional to the speed of the ship relative to the water. Modern mechanical or electrical logs give readings in various parts of the ship, of the speed of the ship and the distance covered by it since the log was set. The passage of the ship through the water rotates vanes which communicate with an instrument not unlike the speedometer of a car. Other patterns depend on the motion and pressure of water through a tube (the pitot tube used in aerial navigation relies on the increase of pressure of air caused by the forward motion of the aeroplane). Rotating-vane types of instrument are also used in aerial navigation.

How to find Latitude

The principle of the sextant should be investigated. A simple sextant to demonstrate the action of the instrument



may be made in the workshop and used to measure roughly the angles subtended at the eye by pairs of distant objects, and to find the 'altitude' of the sun. As the sun is so far away it is assumed that all its rays are parallel when they reach the earth. The altitude of the sun at a place is the angle made by the sun's rays to a tangent plane drawn to the earth at that point. If the sun is overhead at the Equator it will be seen from the diagram that its altitude at noon at any place is equal to $(90^\circ - \text{latitude})$.

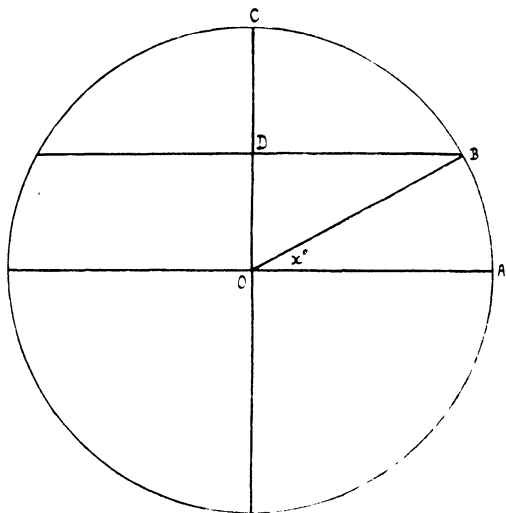
Thus $\text{latitude} = 90^\circ - \text{sun's altitude at noon}$. However, the sun is only overhead at the Equator twice in the year.

During the winter months the sun's declination (obtainable for any date by consulting *Whitaker's Almanack*) must be subtracted from the expression $(90^\circ - \text{sun's altitude at noon})$ and in the summer months the appropriate angle must be added. Observations on the Pole Star and other stars may also be used. Demonstrations with a globe and a distant source of light will make the matter clear.

Degrees of Longitude

Divide the arc of the polar meridian between the Equator and the pole into 90 equal parts of 1° each. Through each of

these draw a 'small circle' in a plane parallel to that of the Equator. These are parallels of latitude. In an actual demonstration two or three such small circles will suffice.



DB = radius of small circle at latitude x° North

$DB = OB \cos x^\circ$

$= \cos x^\circ \times \text{radius of the earth.}$

Thus the circumference of a 'small circle' (360° of longitude)

$= \text{cosine of latitude} \times \text{circumference of Equator.}$

A degree of longitude $= 60 \times \cos x$ sea miles.

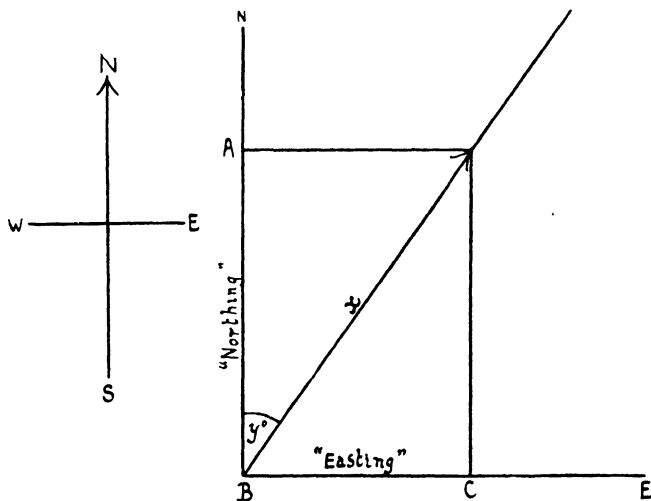
Thus when going East or West

Change of longitude in degrees $= \frac{\text{number of sea miles}}{60 \times \text{cosine of latitude}}$

'Dead Reckoning'

Before the days of wireless-direction signalling, underwater signalling (and even to-day), the sailor unable by climatic conditions to observe the sun and the stars, had to estimate his position by means of 'dead-reckoning'. From

his log-book he knew at what time he last fixed his course, he could estimate his speed by means of the log, and hence calculate the number of miles he had travelled in a certain direction.



Suppose his course set at B is y° East of *true* North (obtain by correcting magnetic North) and the distance travelled is x sea miles (by log).

Then 'northing' $BA = x \cos y$ sea miles

$$= \frac{x \cos y}{60} \text{ degrees of latitude}$$

Easting $BC = x \sin y$ sea miles, or expressed in degrees of longitude $\frac{x \sin y}{60} = \text{cosine of average angle of latitude between A and B.}$

This is called 'middle-latitude reckoning' of longitude. These figures would be added to the bearings at point B. (Corrections must be added owing to the curvature of the earth, but these need not concern us here, especially if x is

not large; and the above process will be repeated a number of times during the whole course.)

Map-Projections

Work on map-projections is definitely for older children, but it is of considerable importance for everyone who uses a map, and gives interesting applications of previous work on mensuration and trigonometry. The problem of representing figures originally on a sphere on flat surfaces is an old one. The type of projections used depends on the purposes for which the flat map is intended. For descriptive maps showing 'distributions' throughout the world the whole globe must be shown on one projection. At other times, a single large land mass such as Africa or Australia is all that is required; or again, navigators in the air or on the sea, surveyors and explorers in the Polar regions need special maps for their specific purposes. It is interesting to think of all the different people who are likely to need maps for their work and to imagine what they will require from them. Some need maps which preserve directions; for others this is not necessary but corresponding areas must be preserved and the possibility of shape distortion is a secondary consideration. All children will know that on the popular Mercator projection, Greenland appears to be larger than Africa, although in actual fact the latter country is at least ten times the size of the former. The danger nowadays is that children see many maps without troubling to refer to the globe. Time spent in examining the globe is never wasted. The school should also possess a large globe of a dull black surface on which the outlines of the land masses are marked in white. Chalk lines such as great-circle sailing-courses can be drawn, and subsequently rubbed out.

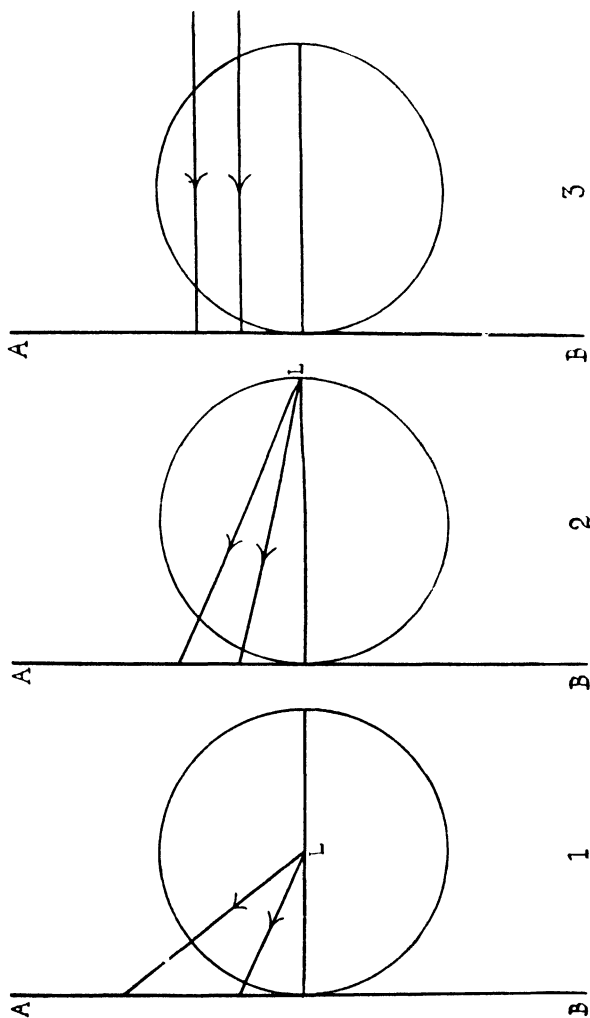
An excellent way of illustrating various types of map-projections is to take a clear glass globe, coat it with gelatine, draw on this a rough outline of the continents and fill in the

land masses with dilute coloured inks (even a fish globe with flat bottom and open top will serve). The room is darkened, a tiny electric bulb is suspended within the globe, and images of the outlined continents will appear on external sheets of paper. The type of projection varies according to the position of the bulb and the position and nature of the surface of the paper—flat, cylindrical, conical. It is often convenient to use a spherical wire cage instead of a globe, as this shows the distortions of the graticule (network) in projecting. Not all the standard projections may be demonstrated in this manner, however.

Zenithal Projections

The three cases opposite, are called respectively gnomonic ('sundial' projection, light at the centre); Stereographic (light at end of the diameter); Orthographic (the light imagined to be an infinite distance away and rays from it parallel). The paper AB is flat. By varying the position of the light between case (i) and case (iii) other projections have been evolved. In the three standard cases given above, the use of simple trigonometrical functions will suffice to show the relation between the projected distances and those on the globe. A simple conical projection is worth demonstrating and although the theory of an elliptical projection (such as that of Mollweide) is a matter of some difficulty the general idea is not difficult to grasp. A circle is first drawn to represent the surface area of one hemisphere. An ellipse is drawn with its minor (smaller) axis equal to the diameter of this circle and its major (greater) axis equal to twice this length. The projection of the whole world is drawn in this ellipse and although the shape of a country is distorted its area is preserved.

Mercator's projection is so largely used for wall maps, atlases and navigators' charts that it should be discussed; although here again the complete theory is a matter of some



difficulty. It is a 'cylindrical' orthomorphic projection, which implies that it is a projection on the inside of a cylinder circumscribing the globe and touching it at the Equator, and that the shape (of a small country) is preserved. The characteristics of the projection are that all lines of longitude and latitude are vertical and horizontal respectively on the projection, and that the latitude scale at any point is equal to that of the longitude scale. Now a small circle of latitude 60° N. is $\cosine\ 60^\circ \times$ the length of the Equator, or just half the length of the Equator. Thus on the Mercator projection the scale at latitude 60° is double that at the Equator. Thus an area in latitude 60° is magnified to four times that of a similar area at the Equator. In the general case the linear scale increases as the reciprocal of the cosine (i.e. secant) of the latitude, and the area as the square of this. At greater latitudes this size magnification increases enormously as we approach the poles where theoretically a small area on the globe is infinitely large. The Mercator projection is thus useless for the Polar regions. It is useful for navigators following the ordinary sailing routes, because all meridians run north and south, all 'parallels' are east and west on a Mercator chart, and a straight line crossing them maintains a constant bearing. Nevertheless, such a straight line does not represent the shortest distance between two points on the earth's surface, which is always a part of a great circle. The great-circle route plotted on a Mercator chart is curved, but it can be followed quite easily by regarding it as a series of straight lines (really chords of the curve), each having a constant bearing.

Navigators using the ordinary sea routes use the Mercator chart for most purposes, but for wireless direction-finding, and air-navigation where the great-circle routes can be readily followed, the gnomonic projection is frequently used, for straight lines on this projection actually represent the shortest distances between two points on the globe.

Children will be interested to learn something of directional wireless telegraphy and new air routes which are constantly being explored, such as that from Moscow to Vancouver which goes practically over the North Pole.

Some further topics which are suitable for lectures for scouts and guides or for science and mathematics societies are: surveying by means of beams of light; surveying by aerial photography, taking pictures at intervals depending on the speed and height of the aeroplane; surveying by photographs taken from a moderate height; modern direction-finding instruments; instruments used at sea; supersonic signalling and sounding (depth-finding); wireless direction-finding; instruments used for aerial navigation and blind flying.

CHAPTER XI

CIVIC AND RURAL ARITHMETIC

The Arithmetic of Citizenship

MANY of the topics dealt with in the arithmetic of citizenship are closely related to civics and history; and the mathematics master should endeavour to obtain the assistance of his colleagues who teach these subjects, in order to avoid overlapping. Usually the arithmetical technique required is slight, but the problems to which it is applied are of such importance in the life of the citizen that they cannot be overlooked.

1. *Banking. Investing Money and Interest*

National Savings Certificates

Their history and rates of interest in the various issues since the War. Saving for certificates by means of stamps. How the certificates are issued.

Post Office Savings Bank

Saving by stamps on cards. Nature of a deposit account; pass book; home safes; where the interest comes from.

Banks

Deposit accounts and pass books. Interest; current account; bank charges and how reckoned; how to fill in a cheque; crossing a cheque; clearing houses; overdrafts. How banks use money which is deposited with them. How interest is earned. How bank rates are fixed (a general account).

Compound Interest

The teaching of simple interest in schools cannot be justified from the utilitarian standpoint, but it is sometimes retained to illustrate certain principles in proportion and to provide a standard of comparison with the more useful (and difficult) compound interest. The 'magic' of compound interest is more easily realized when it is compared with the linear growth of a sum of money by simple interest. Here a graph will be useful.

At 4 % simple interest a sum of money would double itself in 25 years and treble itself in 50 years. At 4 % compound interest money doubles itself in less than 18 years, *quadruples* itself in $35\frac{1}{2}$ years, and is multiplied by eight in little more than half a century.

Compound interest has applications apart from money problems: if we decrease the length of the intervals of time at which the accounts are 'made up', and increase correspondingly the number of intervals of time we get a clue to the 'law of natural growth', which leads to later work on exponentials which have enormous significance in higher mathematics, the growth of living matter, probability, physics and other sciences.

In simple interest, increase of the original sum of money takes the form of adding a fixed sum at regular intervals; in compound interest it may be considered as the repeated multiplication of a sum of money by the same factor at the end of each interval.

Consider the case of £1 invested at 4 % compound interest. At the end of the first year it (the amount) has become

$$£1 \times \frac{104}{100}$$

At the end of the second year it has become

$$£1 \times \frac{104}{100} \times \frac{104}{100}$$

At the end of the third year it has become $£1 \times \left(\frac{104}{100}\right)^3$

At the end of the fourth year it has become $£1 \times \left(\frac{104}{100}\right)^4$

In the case of a Principal of $£P$ and a rate of $r\%$ the amount at the end of the fourth year = $£P \left(\frac{100 + r}{100}\right)^4$

which is sometimes written $£P (1.0r)^4$

It must not be forgotten that interest is found by subtracting the original Principal from the final amount. Such formulæ as the above may readily be worked out by means of logarithms:

In the general case: $A = P (1.0r)^n$ (n is the number of years).

$$\text{or } \log A = \log P + n \log (1.0r).$$

This formula may be manipulated so that if any three of A , P , r and n are known the other may be determined.

E.g., find the number of years necessary for a sum of money to double itself at 3% compound interest.

(Consider a Principal of $£1$ and an Amount of $£2$.)

$$2 = 1 (1.03)^n$$

$$\therefore \log 2 = n \log 1.03$$

$$n = \frac{\log 2}{\log 1.03} = 24 \text{ (nearly)}$$

(*N.B. Divide the logarithms.*)

Ans. 24 years.

This is an example of the method of dealing with simple equations where the unknown is an index.

As the banks 'make up their books' at frequent and regular intervals it is not convenient for them to use logarithmic methods for reckoning the interest, especially

as they often have various other complications such as the deduction of bank charges, withdrawals, deposits, etc. A few examples should be worked to calculate interest over a short period such as $2\frac{1}{2}$ years, the interest being calculated and added directly every half-year. More realistic examples will result by adding deposits and subtracting withdrawals. If the interest is 3 % (per year) it will be taken as $1\frac{1}{2}$ % per half-year, and so on. The long addition method of performing compound interest, though less interesting and more cumbersome than the logarithmic method, is nevertheless more realistic. The logarithmic method is useful in such instances as the following: compare the rates per cent. compound interest offered by recent issues of National Savings Certificates:

15s.	becoming	£1	in	10 years	(6th issue)
16s.	„	£1	in	8 „	(5th issue)
16s.	„	£1	in	7 „	(4th issue)

2. *Stocks*

There are many approaches to the subject of stocks, shares, capital and money in industry. On the one hand we may treat the matter historically, discussing the uncritical manner in which people formerly invested their money in many spurious companies, how money was raised for various great undertakings in the past, the trading companies, the South Sea 'bubble', etc.

The financial pages of the newspapers and the prices quoted at the end of the wireless news-reports will yield much material. The matter can best be understood by a 'play-way' in which a member of the class assisted by two or three others starts a 'Company' with the financial support of the other members who are 'shareholders'; other rival companies spring up, stock-brokers buy and sell shares for their clients; and thus various technical and popular terms

such as 'a corner', 'bull' and 'bear', 'at par', should be explained by citing instances. Children will see how circumstances sometimes arise which give the dishonest a means of gaining money illegally; and what is perhaps more important, how 'sharp practice' which is not in the best interests of society in general has sometimes appeared. Whether we like it or not, financial matters are most potent forces controlling national and international policy, security, trade, the prices of commodities and public well-being; and the cause of democracy is served in a better manner by investigating simple cases of everyday finance, than by mechanically working old-fashioned problems on stocks and shares which have no relation to actual practice.

The procedure at the Stock Exchange in England, at Wall Street in New York, and at stock-brokers' offices will prove interesting and may be illustrated by lantern slides and cinema films. A popular children's game, 'Bulls and Bears', may also prove useful. In some schools children pretend that they have £1,000 to invest, select various stocks, 'buy' them at current prices if they are obtainable, sell them in six months' time and find their total gains or losses.

The arithmetical basis of stocks and shares is not difficult. The price of a stock is given as the number of pounds cash which will buy a hundred nominal pound shares. Often the price is quoted for blocks of shares other than one hundred, or even for single shares; e.g., stock at 75 means that £75 cash will buy 100 nominal pound shares; a 16s. National Savings Certificate for the purpose of discussion may be regarded as a special government stock at 80, etc.

Income is normally reckoned on stock, not on cash; e.g., 3% stock at 75 means that the 100 pound shares bought for £75 cash will yield £3 a year income. Generally speaking, as may be expected, the cheaper the stock the smaller the interest it yields. Gilt-edged or government stocks do not

yield high rates of interest, but they are secure, as they are 'backed by the financial stability of the country'.

When money has been invested in a 'business' which has prospered, owing to public demand for certain commodities and to wise investments, etc., the income yielded is large and the stocks are not only high in price, but very difficult or even impossible to obtain.

Brokerage is reckoned by making the price of the stock dearer to the buyer and quoting it at a lower price for the seller. Examples including brokerage may usually be omitted from school work.

Example. What is my total income obtained by investing £400 in $3\frac{1}{2}\%$ stock at 80 and £600 in 7% stock at 120?

$$\left. \begin{array}{l} \text{Stock obtained by investing} \\ \text{£400 at 80} \end{array} \right\} = \frac{400 \times 100}{80} \text{ pound 'shares'}$$

$$\begin{aligned} \text{Income from above at } 3\frac{1}{2}\% &= \frac{£400 \times 100 \times 3\frac{1}{2}}{80 \times 100} \\ &= £17 \ 10 \ 0. \end{aligned}$$

$$\left. \begin{array}{l} \text{Stock obtained by investing} \\ \text{£600 at 120} \end{array} \right\} = \frac{600 \times 100}{120} \text{ pound 'shares'}$$

$$\begin{aligned} \text{Income from above at } 7\% &= \frac{£600 \times 100 \times 7}{120 \times 100} \\ &= £35 \ 0 \ 0 \end{aligned}$$

$$\text{Total} = £52 \ 10 \ 0$$

Other Topics for Older Pupils

How a Company is formed; Directors and their duties; Shareholders, Prospectuses and Balance Sheets; 'Limited Liability Companies'; Income Tax on Profits and how it is reckoned; Various types of Shares: Preference Shares, Debentures, Ordinary Shares; How Banks lend money to industry; Bank balance sheets. (In each case actual examples

may be taken and discussed.) The finances of Co-operative Societies.

Insurances of Various Kinds

Begin by discussing the history of insurance and why it is necessary.

(a) Fire, 'All-In' and Car Insurances

These may be considered 'historically', and the necessity of such insurances mentioned. The arithmetic is quite simple and examples of the calculation of premiums by simple proportion will be worked. If time permits special insurances, e.g., Lloyds' 'underwriters' and examples of the insurance which they will undertake.

(b) Life Assurance

The origin of life assurance. Expectation of life. Graphs to illustrate how this has increased during the last half-century. A general outline of the methods of estimating expectation of life. Various types of assurance.

- (1) Whole Life Assurance.
- (2) Endowment Assurance.
- (3) Deferred Annuities.
- (4) Special 'family' and other assurances.

Information on these matters may be found in the booklets issued by the various assurance companies.

Calculations may be made to find which are the best types of assurance under particular circumstances and admitting certain risks.

3. National Health and Pensions

Its history and the general benefits effected by it. Cards and stamps. Health, Unemployment, etc., Widows, Old Age Pensions.

Details of the schemes are issued officially and may be obtained at Post Offices, Libraries, Employment Exchanges, or the more comprehensive literature from H.M. Stationery Office.

Other schemes for pensions or superannuation may also be discussed.

Rates and Local Finance

A rate demand notice gives details of the allocation of the money collected, by showing how the rate on every pound of rateable value is divided up. This may be used as the basis of a lesson on rates.

The 'rateable' value of the property in any district is calculated and a 'penny rate' will produce a sum equal to the total rateable value of the district divided by 240. The rateable value of a small house is usually less than its rentable value, as the rents of such houses have increased so much during the last few years. The district or parish rates are divided into two parts, one of which goes to general county expenses and the other for purely local purposes.

Many interesting problems may be made by considering the finances of local government.

A new road or drain may be necessary, the Ministry of Health (from the central funds of the country) will pay a certain percentage (say up to 40%), and the total expense of the undertaking is £30,000 spread over a number of years. By how much (in pence and decimals of a penny) will the rate in the £ be raised? More complex problems will arise when the 'council' borrows money for certain projects, and pays it back gradually together with the interest on the sum still outstanding.

Police, schools, highways, parish relief may be the subjects of interesting problems, and the work should be linked with a 'qualitative' discussion of rates in history and civics classes.

National Finance

This is a good subject for topical consideration when the daily newspapers contain accounts of budget speeches and proposals. A graph showing the rates of income-tax during the last fifty years or so, and another showing the increase of national expenditure during the same period will prove interesting and give a basis for discussion. Some particular points in the story of national finance during the last few centuries are always acceptable. The national balance sheet may be considered in two aspects:

(a) The Income or Revenue of the Nation

Excise: Stamp duties, duties on alcohol, amusements, patent medicines, tobacco, etc.

Customs: Duties on imports: alcohol, tobacco, perfumes, silks and other fabrics, ad valorem duties on many important commodities (*a*) for the protection of home producers, (*b*) for increasing revenue.

Estate Duties: Duties charged as a percentage of the value of an estate when the owner dies or its ownership is transferred. A graph will show how the percentage duty increases with the value of the estate.

Income-tax: Income-tax varies from year to year and the budget proposals should be discussed and simple problems may be founded on them. An income-tax assessment-form and a demand-note may be examined and the income-tax payable under various circumstances of salaries, profits from a business, allowances for wife and children, depreciation, life assurance, etc., calculated. The treatment must be topical.

(b) National Expenditure

Graphs will show how expenditure on the Army, Navy and Air Force, and on the Social Services, Education, etc., have varied since the war. The national debt and sinking funds may be treated historically and graphically.

Financial matters in general may be treated in a concentric manner. The child starts by playing at shops and gains ideas of dividing his small sums of money between present spending and saving for a holiday, a toy, or for a more distant future. It is as unwise to preach a gospel of 'save all' as to recommend 'spend all'. The wise man strikes the mean between spending and saving. The slogans heard so often to-day: 'Spend wisely', 'Save as you spend' are certainly very true, but are not always easy to put into practice. From early experience with the use made of his weekly pence the child may pass, at the end of Primary School life, to a discussion of housekeeping accounts, the allocation of wages earned for various purposes, etc. Many of the periodicals devoted to the home and house-craft give examples of the household budget, which may be considered critically. From this we pass to such topics as the following:

- (1) How to act as the Treasurer of a School Club or Society. The balance sheet.
- (2) The finances of a small shop or business.
- (3) How to make a Poultry Farm pay.
- (4) The cost of a Sunday School Treat or Christmas Party.
- (5) The cost of a Holiday at home or abroad, or of a fortnight's camp.
- (6) The financial aspect of running a car:—cost, tax, insurance, petrol, oil, garage and 'parking', repairs, depreciation, etc.
- (7) Building Societies. Borrowing money to buy a house.
- (8) Hire-Purchase:—advantages and disadvantages, furnishing a house.

Many other examples may be suggested both by teacher and children and an excellent selection is given in the *New Freedom* mathematics books.

A formal treatment of the problems of costing which demands great experience and skill, is outside the scope of school work. Nevertheless, a child may readily glean some ideas about the necessity of costing and how it is done.

Graphs will prove to be very helpful. The boy laboriously engaged at the handicraft centre in the construction of a stool and the girl making a garment should have some idea of the cost of the raw materials used. Add to this 'so much an hour' for time, and the basic cost of the article is known. On a larger scale other factors would enter into the matter:

(1) Rent; rates and taxes; heating; lighting; cleaning; repairing of the building. This is part of the 'Over-head' cost.

(2) If materials were bought in larger quantities or, if it were worth while, made at or near the factory, they would be cheaper.

(3) If many articles are made to the same pattern and a machine is devised to make or help to make the article, time is saved, fewer people are necessary in the factory, and more articles are produced in less time.

(4) Advertisements, agents, salesmen and others who do not actively make the article have to be considered.

Older classes may consider various large manufacturing organizations; and statistics (preferably shown graphically) will serve to show the relation between production and selling prices, the effect of the introduction of machinery, the combining of interests, etc. From a practical point of view some idea of simple costing will be very useful to the pupil who later will work as a smallholder or in other rural occupation.

Mathematics in Rural Districts

Although normal Senior School work precludes any definite vocational training, an agricultural environment with its long tradition, to say the least, will influence considerably the content and treatment of the mathematics course in rural districts. It is obvious that revolutionary changes have been made in farming by the application of scientific discoveries, but what is just as important, successful farms are now run

on an equally sound mathematical basis. Farming has seen such bad times since the War that many farmers have only kept themselves financially solvent by adopting careful means of estimating quantities and values, book-keeping and costing. Many have suffered bankruptcy for failing to do this. A most important feature of the course taken by any student at an agricultural or similar college is the work on 'costing' and related subjects, which should subsequently help him to run his farm or business on an economic basis. It is only by applied arithmetic that a farmer can tell whether he is making the best use of his land from the point of view of financial return. As the farmer pays various taxes, some of which are subject to relief, he often employs an expert to do this work for him. No formal treatment of the intricacies of farm costing will enter the school course, but many simple and useful mathematical problems can be evolved by considering farming and rural life in general.

Some Suggested Topics

Mensuration

The surveying of highways and fields; levelling, determining gradients; measuring perimeters, drawing maps and plans. How these help when extensions and improvements are suggested. Earthenware and other drain pipes. Calculation of areas. (How to find the approximate area of a field with four straight sides by multiplying the averages of opposite sides.) Total lengths of furrows; steam ploughing; question of times and costs. (See also the section on surveying.)

Application of Simple Statistics to Areas

- (1) Number of cattle grazing in a certain area.
- (2) Yield of various crops under certain conditions.
- (3) Time taken for ploughing, sowing, etc.
- (4) Weight and cost of fertilizers employed.

Design of Farm Premises

Plans and sections of outbuildings. Design of byres and other buildings, with a view to requirements of light, cleanliness and convenience. Capacities of tanks, sumps, churns, pits, containers and dams. Length of wire for electrical wiring of buildings. Design and capacity of a Dutch barn. Volumes of stacks, mounds and loads and how to estimate them roughly. Convenient ways of levelling ground so as to move the minimum amount of soil. How to find the heights of trees. Estimation of number of cubic feet of timber in a tree-trunk.

The size and weight of a cheese. Grading eggs. Estimations of weight from volumes. Hay stacks, manure heaps, piles of stones, mounds of earth; applications of circular mensuration; densities, volumes and weights.

Arithmetic

Proportion: Yield of crops; quantities of seed, fertilizer, etc., required for certain fields; hours of work and number of men employed.

Proportional Division: Mixing fertilizers, insecticides and remedies, etc.; allocating money, etc.

Percentages: Percentage germination of seeds; percentage composition of butter, milk, soil, manures, etc. Profit and loss. Farm assurance of various kinds.

Money Problems: Carriage of goods; simple balance sheets; subsidies and concessions; the work of various Boards, e.g., Milk Marketing, Potato, Land Drainage, etc. Calculation of cost of electricity for power and lighting. (A graph will show what are the most economical terms according to the quantity consumed.) Calculation of quantity and cost of various fodders, etc., consumed by livestock.

Older pupils may suggest projects which may be discussed from an arithmetical point of view, such as:

- (1) The finances of a poultry farm; a fruit farm; a pig farm; bee-keeping, etc.
- (2) Is it more profitable to graze cattle, or to farm various root and cereal crops under various sets of circumstances?
- (3) A simple treatment of the arithmetic of Mendelism and breeding.

Graphs

Owing to the ease with which they can be read, graphs are particularly helpful for expressing the statistics of the farm. Graphs may be drawn showing rainfall, yield of milk, of eggs, of other crops, fluctuations in prices, cost of electricity, or other 'power' which is used, etc.

Mechanics

Levers; pulleys (Class II and Weston); mechanical advantage; special types of levers (lifting truck), and pulleys adapted for farm work.

Moments, stresses and strains, roof trusses. Various types of weighing machines and how they work. Mechanical principles of gates, fences, derricks. General idea of a force. Work as force $\times$ distance.

Power as force $\times$ speed or $\frac{\text{force} \times \text{distance}}{\text{time}}$

Horse-power, how it is measured; engines.

Minimum horse-power required for certain tasks, e.g., H.P. given by or required for traction-engine, a tractor, a threshing-machine, a chopper, etc.; water turbines and wheels; 'head' of water; how to estimate power of water-wheels. The inclined plane and screw.

Centrifugal force: the milk separator; threshing-machine, etc. The action of standard farm machinery: the plough, rake, drilling and mowing-machine, 'self-binder', stacking-machine, milking-machine, cheese-press, etc. This

may be combined with the rural science course, and it should endeavour to show how simple science used quantitatively will help to give efficient working.

Densities: Home-made hydrometer for milk testing.

The publications of the Ministry of Agriculture and the schemes of the Milk, Potato, Pig, Drainage Boards, etc., contain data which may be converted into excellent practical problems in simple mathematics. Periodicals such as *The Farmer and Stockbreeder* and those dealing with the keeping of poultry and other livestock are often concerned with the financial and quantitative aspects of the matters with which they deal, and many of the examples they give may be turned into useful exercises or suggest interesting projects.

CHAPTER XII

SOME BYWAYS OF MATHEMATICS: PROPERTIES OF NUMBERS,
CALCULATING MACHINES; HISTORY OF MATHEMATICS;
MATHEMATICAL SOCIETIES

Properties of Numbers and Puzzles

MANY puzzles, and some of the simpler properties of numbers have for centuries been an interesting relaxation from strenuous forms of mathematical work. Indeed, for nearly 5,000 years the 'magic' properties of numbers have been a source of wonder, and have often borne great mystical and religious significance.

The first and last books of the Bible contain references to numbers which were imagined to hold important secrets. 'Researches' to find the 'meaning' of the number 666—the 'mark of the beast', have occupied whole lifetimes, and the results would fill many books. Just as the quest for the Elixir of Life or the Philosopher's stone brought many useful discoveries in science, so the pursuit of number relations for their own sake has not been without its effect on orthodox mathematics. Many interesting topics in the history of mathematics can be found concerning the mystical and magical 'properties' of number and formulae. Even so stable-minded a physicist as Heinrich Hertz (the discoverer of 'wireless waves') said only half a century ago, 'One cannot escape the feeling that mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than was originally

put into them.' He was, of course, referring to the immense use to which formulae can be put in the sciences and their applications and not to any so-called 'magical' properties.

It would be difficult to justify the spending of much time on the 'theory of number' or mathematical puzzles, from the purely utilitarian point of view; though it is surprising that seemingly 'useless' sets of numbers have recently been applied to branches of science so far removed as genetics and 'wave mechanics'. From the child's point of view an investigation of the properties of number assists in the process of 'making friends with numbers', and it can form the basis of a fascinating pastime. No work with numbers should be despised, for ability to factorize, decompose, square or find a square root quickly, is of great assistance to many of the processes of calculating.

Children are always interested in magic squares and cubes, number cross-word puzzles and number codes.

Magic Squares

4	9	2
3	5	7
8	1	6

Find some other simpler ones

1	15	14	4
12	6	7	9
8	10	11	5
13	3	2	16

(The numbers in each column, row and diagonal add up to 34.)

A Magic Cube (distorted in shape to show the numbers)

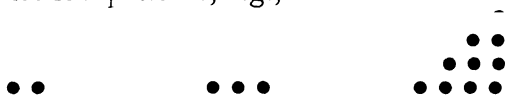
10	26	6		24	1	7		8	15	19
23	3	16		7	14	21		12	25	5
9	13	20		11	27	4		22	2	18

(Find how many sets of 3 numbers in a straight line add up to 42.)

It will be interesting to hunt for some further examples.

The construction of series of numbers is also acceptable to Senior School children in the later forms. With very little help children will deduce some of the properties and possible uses of the series.

As is well known, the Greeks excelled in geometry but not in arithmetic and algebra. Their sets of numbers were founded on geometrical patterns, e.g.,



Triangular Numbers

Other sets may be made from squares, pentagons, 'stars', hexagons, and more complex patterns. (Hogben's chapter, 'The beginnings of arithmetic', contains many examples of this type of number.)

The sieve of Eratosthenes and various ways of testing divisibility and finding factors are always fascinating; tests

of divisibility by 2, 4, 3, 5, 9, and in special cases of 11, 37, are useful and interesting.

The powers of 11:—121, 1331, 14,641, etc., anticipate the binomial theorem; indeed, attempts have been made to explain the well-known exponential 'curve of error' through these numbers. Arranged in various forms they give Pascal's and other related triangles.

Some of the simple number theorems are easy to prove, whilst others are of the utmost difficulty, and in some cases a general proof has not yet been found:

(1) Any odd number can be expressed as the difference of two squares.

$$\begin{aligned} 3 &= 2^2 - 1^2 \\ 5 &= 3^2 - 2^2 \\ 7 &= 4^2 - 3^2 \\ 9 &= 5^2 - 4^2 \end{aligned}$$

The law is easily seen—divide the odd numbers as nearly as possible into two halves, e.g., 37 divides into 18 and 19. $37 = 19^2 - 18^2$. (Square numbers and square roots can be found by applying Pythagoras' Theorem.) Simple algebraic formulae will serve to prove the general case, e.g., $(n+1)^2 - (n)^2 = 2n+1$ (which is always odd if 'n' is an integer).

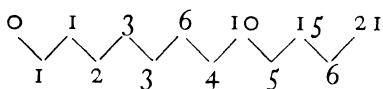
(2) Every square number ends in 0, 1, 4, 5, 6, or 9.

(3) All cubic numbers can be expressed as the differences of two squares.

$$\begin{aligned} 1^3 &= 1^2 - 0^2 \\ 2^3 &= 3^2 - 1^2 \\ 3^3 &= 6^2 - 3^2 \\ 4^3 &= 10^2 - 6^2 \\ 5^3 &= 15^2 - 10^2 \end{aligned}$$

The series of numbers in the vertical columns at the left

are interesting and will give a clue to the general formula:—



Expressed algebraically: $\left\{ \frac{n^2}{2} + n \right\}^2 - \left\{ \frac{n^2}{2} - n \right\}^2 = n^3$

(4) *Waring's Theorem*

That every integer is either a square or the sum of 2, 3, or 4 squares. Perform experiments with simple cases, e.g., $46 = 5^2 + 4^2 + 2^2 + 1^2$.

(The general theorem for any higher power, e.g., using 9 cubes, 19 4th powers, etc., has only recently been proved.)

(5) *Goldbach's Theorem*

'Every even number is the sum of two primes.' The general theorem has not yet been proved. Children will write down any even number up to 200 and later to 1,000, and then hunt for the primes. Children will be interested in the 'perfect' and 'amicable' numbers of the ancients. A perfect number is equal to the sum of all its factors including 1, e.g., 6, 28, 496.

$$6 = 3 + 2 + 1 \quad 28 = 14 + 7 + 4 + 2 + 1$$

Pairs of 'amicable' numbers are such that each is equal to the sum of all the factors of the others, e.g., 220 and 284 are such a pair.

Number acrostics, cross-number puzzles and similar things are also interesting.

A list of books on these and other related topics will be found at the end of the last chapter. It is not suggested that whole lessons should be given to the topics mentioned above, but a few minutes occasionally devoted to these and similar 'puzzles' will stimulate interest and bear fruit later.

Calculating, Recording and Counting Machines and other Instruments

Children are always interested in 'things which work', and although many of the modern calculating instruments are of extreme complexity the simpler types are of such widespread use that no teacher of mathematics should fail to learn something about them, and give his pupils the opportunity of examining and using them. If we can regard such instruments as sets of cylindrical Napier's bones as calculating machines (or even going much further back than this to the Ball-frame or Abacus), it is evident that instruments for calculating are of great antiquity. In fact, if we are not too particular about the use of the word 'machine', we may say that 'machines' were used because very few people could calculate in the simple mental and 'paper-pencil' way; just in the same way that many churches in England had barrel-organs of quite complex construction long before instruments with ordinary keyboards superseded them. The very word 'calculate' (calculus—a stone) shows that stones and pebbles must, in a way, have been the precursors of all instruments for calculating; and by developing this the subject may be approached historically, mentioning such things as Napier's bones and Pascal's and Leibniz's adding-machines.¹ The invention of the 'pin-wheel' was a veritable milestone in the history of the adding machine. Multiplication could be performed as a series of repeated additions, for it enabled the effect of the total of previous numbers to be passed on.

From the practical point of view it would probably be better to start by considering clock-work mechanism for counting purposes, and reducing gears. Several useful pieces of apparatus can readily be acquired second-hand, e.g., a 'mile-o-meter' or cyclometer, such as is attached to the front wheel of a boy's bicycle or a car 'speedometer' which,

¹See Wolf, *History of Science and Technology*.

besides measuring speed, will also register total distance and probably the distance travelled on a particular journey. Not only should the principle of the mode of working of these simple instruments be investigated, but it will be interesting to calibrate them under varying conditions and to endeavour to foresee where errors are likely to appear, as in the case of using worn or partially deflated tyres on the bicycle or car, and to estimate these errors.

Other devices employing a train of wheels are to be found on domestic electric watt-meters, gas meters, water meters, devices for counting telephone calls and at turnstiles. The two former instruments are frequently supplied to schools for demonstration purposes at cheap rates. It is interesting also to investigate the reducing mechanism of a good clock, and to find the ratios of the numbers of cogs and pairs of wheels. Gears such as are used on bicycles, motor-cars, lathes and steam turbines can be investigated in the same way. In industry, automatic counting and grading instruments which are often associated with very ingenious applications of the photo-electric cell, are both exceedingly useful and interesting to investigate.

Many instruments are capable of drawing a graph which records the variation of some quantity with time. The continuous-reading watt-meter and voltmeter are well known to visitors to electric power-stations. Probably the most accessible instrument of this class for class-teaching purposes is the continuous reading aneroid barometer. This can be applied with advantage to illustrate the teaching of graphs, as the idea of variation with time is shown excellently. Machines of this pattern are used for recording tides, earthquakes, rainfall, temperature, humidity, or indeed, almost any quantity which changes with time, and of which it is desirable to keep a record for the purpose of increasing efficiency, future guidance or even scientific interest. All observatories make great use of graphs with time abscissae.

Calculating machines have simplified many operations in business, which were formerly very tedious and liable to errors. Banking establishments, co-operative society offices where thousands of small sums representing individual purchases have to be totalled, and indeed all organizations where many sums of money have to be considered now use comptometers and book-keeping machines. The National Cash-Register, which has been the starting-point for the development of much more complex instruments, is also interesting. The preparation of such volumes as the important *Nautical Almanack* is now greatly simplified by the use of such calculating machines.

We have already mentioned the compounding of sine curves of 'wave-lengths' in the harmonic ratios 1, $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$, etc. (or frequencies in the ratios of the natural numbers 1, 2, 3, 4, 5, etc.) to give complex wave-forms; and conversely, the analysis of complex recurrent forms into simple harmonic sine curves. There are useful machines which will perform these operations, known as harmonic synthesis and analysis respectively. These may be regarded as instruments which put into practice Fourier's Theorem, the basic idea of which is not difficult to grasp. By means of a single band passing over pulleys, a number of sine-movements of different frequency, amplitude (distance of swing to and fro) and phase (starting-point or position at a given instant with respect to one another) can be combined. This is very useful for calculating tides, and for many astronomical calculations. Machines for performing the reverse process (analysis) are of use for splitting up complex musical tones into their harmonic components, etc.

There are also machines for drawing maps and figures to predetermined scales. These pantographs and eidographs are usually quite easy to understand, and can be constructed very simply in the workshop or from Meccano parts. Planimeters are used for measuring enclosed areas, and are

easy to work, though the theory of their action sometimes calls for fairly difficult mathematics. There are also machines for drawing and reproducing complex patterns, lathes for cutting out elaborate shapes in the forms of mathematical curves, simpler devices for drawing curves on paper, and instruments of great complexity for performing integrations and solving equations. As scientists and mathematicians are constantly requiring the solutions of equations and the values of integrals, which would normally require months of labour, such machines lighten their labours and produce the results more quickly. Laboratories containing calculating machines are being set up in various parts of the country, notably at Cambridge. Another reason for the necessity of such instruments is that so many things in life are now treated statistically. A glance at the various lists of figures issued by the Registrar-General, the various Government Departments, economists, biologists and indeed all those who apply modern metrical methods to their problems, will show how colossal must be the task of so much computation. Here calculating machines are of the greatest assistance. At the time of the last general census much interesting information was available about the tabulating machines which dealt with the estimation of the numbers of various factors given on the forms, through the medium of punched cards. Teachers and children will be interested to visit the section in the South Kensington Science Museum dealing with calculating instruments and related devices. Here, in addition to the simpler machines, will be seen a Kelvin Harmonic Analyser, and the results of Babbage's long and costly toil to produce a 'difference engine'. Even if a thorough appreciation of the action of all but the simplest machines which are used for calculating, is beyond the grasp of the children, they will be interested to see mathematics being put into practice; and they may even be led to explore the matter further in after-school life. An automatic telephone ex-

change, a totalisator, an averaging machine, etc., will form the basis of many interesting investigations in later years.

The History of Mathematics in Schools

No study more than mathematics needs considering in terms of its history, and yet no subject has had this aspect of it neglected in education to such an extent. University first-degree courses have found little or no time for it, though it is pleasing to think that it is a possible topic for post-graduate courses. Secondary Schools have tended in the past to take their lead from the Universities, and as their courses have been controlled by academic examinations, no time could be found for what was unfortunately regarded as an optional study of small importance.

Through its history, mathematics is revealed as a dynamic progressive subject, full of human interest and finding its development through its applications to those things which make life fuller and richer. In the physical sciences the widely applied cardinal discoveries are of much more recent origin than those of mathematics.

For instance, Faraday's discovery of electro-magnetic induction, which was the foundation of the greater part of all subsequent work on electricity, is easily demonstrated to any intelligent Senior School child, but the work of Faraday's mathematical contemporaries is, for the most part, of the utmost difficulty even to sixth-form boys and University Students.

This is perhaps one reason why the history of mathematics has suffered neglect. The mathematical inventions, which may be regarded as equivalent from the utilitarian point of view to some recent discoveries in inductive science, often came so early that we are apt to think that they have always been used by mankind.

(1) Many subjects may be introduced by discussing their history, and we have already indicated how this may be done

in trigonometry and algebra. The 'theory of recapitulation', by indicating the parallelism between the general development of knowledge historically considered and its particular development in the ever-growing attainments of the child, may give a clue to the difficulties which he is likely to meet and also supply examples which will prove acceptable to him.

(2) The history of mathematics supplies many instances of solutions to practical problems. It reveals the essential usefulness of mathematics and shows that at every stage its main developments were due to man's needs. We have already seen how arithmetic grew up largely as a result of man's need to assess his wealth, geometry developed from the surveying of land, logarithms and trigonometry were tools to assist astronomy which in turn was the servant of navigation. Again, certain problems in conics and solids of revolution which might appear to be largely academic in interest were suggested to Kepler by investigations to find accurately the volume of a barrel, during an exceptionally good wine year (1612) in Austria. Commercial necessity has often been the mother of mathematical invention. Many of our mathematical terms and conventions can only be properly understood by reference to their history.

(3) An entire lesson devoted to the history of mathematics with suitable pictorial and blackboard illustrations provides a useful change from the usual types of lesson—it links the subject with general history, it infuses life and humanity into the austere symbolism of mathematics. Instead of the 'eternal truths' of mathematics unfolding themselves in logical sequence on the pages of a text-book, one gains the impression of men struggling with clumsy and unrefined methods; and in this the child has something akin to a fellow-feeling. The subsequent history of mathematics, particularly of text-book mathematics, has often been to strip the flesh from these early researches, to remove them

from their utilitarian and philosophical contexts, to arrange them in artificial order with the idea of developing a self-contained system complete in itself but otherwise glorying in its splendid isolation. 'The very crudities of the first attack on a significant problem by a master are more illuminating than all the pretty elegance of the standard texts which have been won at the cost of perhaps centuries of finicky polishing' (E. T. Bell, *The Queen of the Sciences*).

Mathematics in its historical setting appears as a living, practical and human subject, but this is not all. It has far-reaching connections with philosophy, science and other branches of knowledge, and it is well to gain the impression that mathematics is not an isolated subject. Children like to read or hear about people and what they thought and tried to do. The history of mathematics reflects the aspirations and work of mankind.

The 'topic' method will be most useful in the early stages and no attempt can be made to give a continuous account of the development of the subject during the last 4,000 years.

A few topics in the History of Mathematics:

- (1) Early methods of counting; the abacus; notation; Roman numerals; origin of various terms; calculation, mathematics; algorithm.
- (2) Arithmetic in the Bible.
- (3) Geometry in China and Egypt.
- (4) The origin of algebraic symbolism, logarithms, fractions, decimals.
- (5) The life and work of Archimedes.
- (6) Mathematics in the time of the Greeks; its relation to other subjects.
- (7) Astronomy in the time of the Greeks; Astronomy since the Renaissance.
- (8) Early methods of surveying; use of cross-staff (baculum) and astrolabe (theodolite).
- (9) The Hanseatic League and commercial arithmetic.
- (10) The story of calculating machines.

- (11) The history of money, weights and measures.
- (12) The story of the calendar.

A fairly comprehensive list of works on the history of mathematics is given at the end of this book. Of particular value where a 'topic' method is used is Smith's *History of Mathematics*, Volume 2.

Mathematics Societies

There is no reason why mathematics societies should not prove as popular and successful as those devoted to geography or science. Mathematics has so many interesting byways (for which time can hardly be found in school hours) that after-school lectures or demonstrations in the winter, and camps, school journeys, and visits to museums and 'works' in the summer will prove to be of great interest and utility.

Work may include lectures illustrated with book plates projected by the episcopes, or lantern slides and cinema films; or again, experts may be induced to demonstrate calculating and astronomical instruments. Below a few of the many hundreds of topics which may be discussed in mathematical societies are given:

- (1) How the comptometer works; other calculating machines.
- (2) Tides and how they are calculated.
- (3) Making and testing precision instruments.
- (4) Patterns made with the harmonic pendulum.
- (5) Automatic telephone exchanges. How they 'get the right number'.
- (6) Clocks and their history. How time is kept accurately.
- (7) Early instruments for surveying and navigating.
- (8) Will it happen? First ideas about probability.
- (9) Surveying a new country; surveying from the air.
- (10) Magic squares and cubes. Number puzzles.
- (11) Primes and their properties.
- (12) How the distances of stars are measured.

- (13) The life and work of a great mathematician or astronomer, e.g., Archimedes, Stevin, Galileo, Newton.
- (14) Curves and the cathode-ray oscillograph.
- (15) The arithmetic of patterns and curves.
- (16) Biology and mathematics.
- (17) Life Assurances.
- (18) Blind flying. Wireless direction-finding.
- (19) Air-routes of the future.
- (20) A night in an observatory.
- (21) How earthquakes are measured and the position of the disturbance calculated.

CHAPTER XIII

MATHEMATICS ROOMS AND LABORATORIES; APPARATUS AND OTHER AIDS TO THE TEACHING OF THE SUBJECT; MATHEMATICS LIBRARIES

It is a great convenience in Post-Primary Schools to have a large room which can be set aside for mathematics teaching. Obviously, it will not be possible, or necessary even though it were possible, to use it for every mathematics lesson which is given in the school. Some Central and Modern Schools find that in teaching mathematics they are able to adopt a partial Dalton Plan ('assignment', or 'laboratory' plan). For some of the work the pupil is given an assignment which will occupy him for a week or a fortnight. Thus, his speed of work is not compromised by that of quicker or slower boys in the same class. A mathematics room is particularly useful under these circumstances. Such a 'laboratory' should be as well illuminated as is an art room, but it is a great convenience to have an efficient means of darkening the room, for many useful demonstrations in mathematics (quite apart from illustrative work with the optical-lantern, epidiascope or cinema) can be performed by means of lights and shadows, e.g., showing the nature of the various geometrical and map-projections, the phases of the moon, and surveying on a small scale, by means of points of light. A small ante-room may be considered a luxury, but the 'laboratory' itself should be provided with shelves and cupboards for the storage of apparatus, which may well relieve the congestion in the science department by taking over some of the instruments used for elementary physical measurements. Apart from precision instruments (which are not often required for Senior School work in practical

mathematics) most of the apparatus can be constructed very simply in the school workshops. Roof struts and a single water tap and sink are desirable. Mathematics rooms require ample blackboard space, and it is worth while to have one or two blackboards ruled off horizontally and vertically in white lines an inch apart, with every tenth line made thicker than the others, for graph drawing. The writer modified one of these by boring $\frac{1}{8}$ -inch holes at each point of intersection of the thick lines, and the demonstration board so produced was found to have innumerable uses. Triangles of forces, pulleys, moments and other experiments could readily be performed with its aid; and by attaching a nail to the end of a metre scale and inserting the nail through one of the holes in the centre of the blackboard, a 'radius-vector' for generating angles is easily produced. This was also used for drawing circles rapidly. Plywood circles, divided into 36 sectors, could be revolved about a nail placed in one of the holes near the left side of the board, and used as generating-circles for plotting the graphs of trigonometrical functions. The list of apparatus given below may prove useful in Senior School work. In the case of apparatus for teaching mechanics, expensive models are merely a waste of money, as the experiments should be performed with real things on as large a scale as is possible. The following are always useful: Ropes, chains, wheels and axles of various kinds, Meccano parts, a small electric motor, a few cheap balances, a large balance, tins, packing-cases of many kinds, a good large globe with dull black surface and the continents faintly outlined in white, lengths of dowelling fitted into blocks of wood, cotton reels, drawing-boards, metre scales, miscellaneous blocks of wood, brick, stone, metal, stout wire, cord, thread, sheets of tin-foil and cardboard. Often the first-year general physics course can with advantage be undertaken by the mathematics teacher.¹

¹See the Section on equipment and apparatus in the author's *The Teaching of Science*.

1. Slide-Rules. These, made of stiff card, may be purchased from educational supply firms for a few pence each. Children may improvise them by fixing with paste two strips of logarithmic paper side by side, on the top of an old-fashioned pencil-case with sliding cover. One piece should be fastened to the sliding cover and the other to the fixed top. A large wooden model of a slide-rule, graduated as simply as is possible, may be made in the workshop.

2. Surveying apparatus, a plane-table, a prismatic compass, some stands or tripods, a clinometer, a simple theodolite, ranging-rods, a chain, dowels cut into yard lengths, six-foot scout-staves marked in feet and each alternate foot painted red, some tent or other pegs, a supply of large protractors.

3. A model sextant; a wooden base with two plane mirrors for demonstrating range-finding.

4. Models of old astronomical or nautical instruments which can be made in the handicraft centre, e.g., an astrolabe; a baculum.

5. A supply of hollow objects in thin metal; spheres (with and without small hole); cone; pyramid; tetrahedron; cube; prism; oblong box; trapezoid; a quantity of coarse shot or ball-bearings; and marbles to illustrate 'packing', piling and various 'two-dimensional' numbers. The various standards of measurement; and an assortment of common objects, such as buckets, petrol tins, cups for experiments in capacity; test-tubes; graduated cylinders; beakers; spring balances, both of the extension and compression varieties, for work on density and specific gravity.

6. The mechanism of a pendulum clock; a home-made electrically-operated pendulum, etc. A wheel mounted in bearings to show gyroscopic action. Models of various types of weighing machines.

7. The usual apparatus for first-year work in general physics, most of which has already been mentioned.

8. Systems of levers used for reducing maps. (Eidograph or Pantograph); a simple planimeter of the polar type, a harmonic pendulum; a 'pin-wheel and cam disc' model to illustrate the action of an adding-machine; an opisometer (tracing-wheel), a cyclometer; an old speedometer, etc.

9. A set of jointed wooden rods for showing stresses, strains and moments in models of roof-trusses, bridges, etc.

10. Wire models of various solids.

11. Metre-scales, a large model vernier, models illustrating the screw-gauge and the spherometer.

12. Various simple mechanisms, a gear-box, a four-stroke engine, a Watt indicator, etc.

Illustrations

Much useful illustrative material can be made in the form of wall charts for use in mathematics rooms. These may be changed at intervals. From the early days of the Primary School, children will learn a considerable amount, entirely without effort, by merely looking at wall charts in their spare moments. In the early days these will take the form of coloured number boards, various simple shapes with the names boldly printed, coloured circles to represent fractional division, weights, measures and time. Many school charts (in all subjects) fail in a large measure because they sacrifice simplicity, boldness and directness to a desire to make the chart a *multum-in-parvo*, condensing on it too great an amount of material.

In the Senior School wall charts may take some of the following forms:—

1. Charts representing the various epochs in the history of mathematics (some good examples are published by the University of London Press).

2. Portraits of great mathematicians, e.g., Newton, Galileo, Stevin, Einstein, Pascal, Laplace, Leibniz, Bessel, Maxwell, Hamilton, etc. (In the case of those who lived in

more recent times, only the most general indication of their work will be appended to their portraits.)

3. Charts representing various 'scenes' from the history of mathematics, e.g., the discoveries of Thales with regard to triangles; how Eratosthenes measured distances on the earth; Pythagoras' Theorem; (Newton and the apple might well be avoided); Galileo and the pendulum.

4. Surveying in difficult areas. Pictures and diagrams of surveying by photographs. Various types of map-projections. Examples of a simple survey.

5. Various curves and graphs and how they are drawn. Their use and application. The Harmonograph.

6. Astronomy. Pictures of large telescopes, and what they reveal to us. Nebulae, Mars, the moon, Jupiter and its satellites, Saturn's rings, etc. Constellations. Tides and how they are produced. Sine curves compounded. An old astronomical clock showing date, time, changes in the moon, hours of daylight, twilight, darkness, etc.

7. Mathematical machines and instrument..

8. Diagrams showing geometrical relations, and their applications.

9. Statics. Stresses and strains. Simple structures. Bridges. Weighing machines.

It must be stressed that this list is by no means exhaustive and that charts should not be made to take the place of experiments and demonstrations with real apparatus, where this can be obtained. Lantern slides and book diagrams projected in an epidiascope make an acceptable change in a mathematics lesson. The lantern can also be adapted for showing various types of map and plane projections. The use of shadows in teaching geometry has not yet been fully exploited. Suitable illustrative material will be found in some of the smaller works on the history of mathematics, in Hogben's *Mathematics for the Million*, in Wolf's *History of Science and Technology*, in various general works which use

pictures freely for 'self-educational' purposes, etc. The Science Museum at London will supply post-cards of most of the models on view. If mathematics teachers are able to visit the many museums devoted to science, mathematics and related subjects which exist throughout Germany, they will find a wealth of suitable material in post-card form. The Bureaus of Standards, Weights and Measures and similar authorities in most countries issue interesting pictorial material.

Cinema Films

'If we ask what true helps exist we need not look far afield for new materials. When the educational powers of the cinema have been adopted to visualizing the use of mathematical symbols anything which can be achieved by books and blackboards will seem trivial by comparison'—Professor Hogben. At the time of writing this book not a large number of mathematics films have been made and many of these are disappointing. Some of the films illustrating the history of mathematics and mechanics are very interesting, but hardly superior to a set of good lantern slides on the same topics. Films which show mathematical machines, various movements of mechanical instruments, the composition of velocities, accelerations, planetary motions, tides, astronomy, curves and the general 'dynamic' aspects of mathematics and their applications are interesting and useful; though in the case of machines and movements, practical experiments are in most cases better than the second-hand presentation of the screen, but it must be remembered that the cinema film can have complete control of the time-ordinate either by increasing or decreasing its scale. This property is being explored in biology but there is still the entire field open in mathematics and physical science. As there is no projection planetarium in this country (although there are at least three excellent instru-

ments in Germany), stellar, lunar and planetary movements shown by means of a film, are an excellent adjunct to work with an improvised orrery.

Films have also been made to illustrate the simple laws of probability, curves of error, surveying and its difficulties in mountainous countries, range and direction-finders, seismographs and methods of finding the centre of disturbance, micro-measurement and gauge-testing, the solution of quadratic equations, the meaning of simple differential equations and how to solve them, differentiation and integration and their applications, harmonic motion, analysis of curves, tides, compound harmonic motions, compound pendulum and harmonic patterns. However, the majority of these topics is outside the scheme of Modern school work.

It is impossible to say what the future may bring, even though present films do not yield much that is directly helpful to teachers of mathematics. Nevertheless, they should lose no opportunity of examining and criticizing any new films which appear from time to time.

Mathematics Libraries

It is appropriate that the mathematics room should contain a library of books including standard text-books, works of reference, tables, almanacks, charts, and what are more important from the pupil's point of view, popular or simple works on the byways of mathematics and its related subjects. Besides studying readable books on mathematics, its history and applications to our lives and to the various branches of science, pupils should be encouraged to seek for the uses of mathematics in a much wider field.

Many travel stories make considerable use of arithmetical calculations where they describe the measurement of distances and time; and in the process of discovering the criminal, detectives have to estimate the amount of petrol

consumed, the speed travelled, the time that has elapsed since the crime, etc. Although they have lost their youthful freshness, some of the stories of Jules Verne and E. A. Poe contain interesting mathematical allusions; and in more recent times Mr. H. G. Wells treats some of his scientific stories from a mathematical point of view.

The following books, as far as is possible, have been arranged in each list in order of difficulty, commencing with the simpler works. Many of them are only intended for the use of the teacher.

BOOK LIST

ARITHMETIC IN THE NURSERY, INFANT AND PRIMARY SCHOOLS

- The Teaching of Arithmetic*: Monteith (Harrap).
The Groundwork of Arithmetic: Punnett (Longmans).
The Psychology of Arithmetic: Thorndike (Macmillan).
The Case Against Arithmetic: Renwick (Simpkin).
Number Land: Bentzon and Hjuler (Dent).
Number in the Nursery and Infant School: Kenwick (Kegan Paul).
Infant and Nursery Schools (pp. 135 *et seq.*) (H.M.S.O.).
The Primary School (pp. 175 *et seq.*) (H.M.S.O.).
Suggestions for Teachers 1937 Edition. (pp. 499 *et seq.*) (H.M.S.O.).
Mental Tests: Ballard (University of London Press).
The Psychology and Teaching of Number: Drummond (Harrap).
The Montessori Method: Montessori (Heinemann).
The Advanced Montessori Method, Montessori (Heinemann).
For Backward Children:
The Backward Child: Burt (University of London Press).
The Education of Mentally Defective Children: Descocudres (Harrap).

SENIOR AND MORE ADVANCED SCHOOLS

- Mathematics in Senior Schools* (B. of E. pamphlet No. 101) (H.M.S.O.).
The Education of the Adolescent (The Hadow Report, pp. 214 *et seq.*) (H.M.S.O.).
The Teaching of Algebra and Trigonometry: Nunn (Longmans).
A Study of Mathematical Education: Branford (Oxford).
Exposition and Illustration in Teaching: Adams (Macmillan).
Craftsmanship in the Teaching of Elementary Mathematics: Westaway (Blackie).

Many publications of the Mathematical Association (Bell) deal with the teaching of Mathematics at this stage.

GENERAL WORKS ON THE NATURE AND SCOPE OF MATHEMATICS

- The Meaning of Mathematics*: Brodetsky (Benn).
An Outline of Modern Knowledge, Chap. V: Rice (Gollancz).

An Introduction to Mathematics: Whitehead (Thornton Butterworth).

The Nature of Mathematics: Jourdain (People's Books (O.P.)).

Mathematics for the Million: Hogben (Allen and Unwin).

Number, The Language of Science: Dantzig (Allen and Unwin).

The Queen of the Sciences: Bell (Williams and Wilkins).

Mathematics in Life and Thought: Forsyth (University of Wales Press).

The Nature of Mathematics: Black (Kegan Paul).

Introduction to Mathematical Philosophy: Russell (Allen and Unwin).

THE HISTORY OF MATHEMATICS

The Story of Arithmetic: Cunningham (Allen and Unwin).

History of Mathematics in Europe: Sullivan (Oxford).

The Great Mathematicians: Turnbull (Methuen).

Archimedes: Heath (S.P.C.K.).

The Copernicus of Antiquity: Heath (S.P.C.K.).

Galileo: Bryant (S.P.C.K.).

Kepler: Bryant (S.P.C.K.).

Euclid: Frankland (Newnes).

Sir Isaac Newton: Pullin (Benn).

Sir Isaac Newton: Brodetsky (Methuen).

Sir Isaac Newton (ed.) Greenstreet (Bell).

Five Wall Charts: Edwards (University of London Press).

1. The Beginnings of Mathematics.
2. Egypt, Babylon and China.
3. The Mathematics of Ancient Greece.
4. The Dark Ages; The Arab Schools.
5. Beginnings of European Mathematics; Modern Mathematics.

An Introduction to the Study of the History of Mathematics: Sarton (Harvard University Press).

The Teaching of Arithmetic through 400 Years: Yeldham (Harrap).

A Short History of Mathematics: Sanford (Harrap).

John Napier and the Invention of Logarithms: Hobson (Cambridge).

Ten British Mathematicians: Macfarlane (Chapman and Hall).

Pioneers of Science (Astronomers): Lodge (Macmillan).

A Short Account of the History of Mathematics: Rouse-Ball (Macmillan).

- A History of Mathematics*: Cajori (Macmillan).
Men of Mathematics: Bell (Gollancz).
Number, the Language of Science: Dantzig (Allen and Unwin).
British Weights and Measures: Watson (Murray).
A History of Mathematics (2 vols.): Smith (Ginn).
A History of Greek Mathematics: Heath (Oxford).
History of Science and Technology: Wolf (Allen and Unwin).
Science in Antiquity: Farrington (Home University Library).
World Machine: Snyder (Longmans).
Search for Truth: Bell (Reynal).

MATHEMATICAL PUZZLES AND AMUSEMENTS

- Stories about Number Land*: Ponton (Dent).
Stories about Mathematics Land: Ponton (Dent).
Amusements in Mathematics: Dudeney (Nelson).
Canterbury Puzzles: Dudeney (Nelson).
Modern Puzzles: Dudeney (Pearson).
Puzzle Papers in Arithmetic: Boon (Mills and Boon).
Companion to School Mathematics: Boon (Longmans).
Mathematical Recreations and Essays: Rouse-Ball (Macmillan).
Number Games and Number Rhymes: Smith (Columbia Press).
Number Stories of Long Ago: Smith (Ginn).
The Puzzle King: Seward (Lawley).
Brush Up Your Wits: Phillips (Dent).

SURVEYING, MAP-PROJECTIONS, NAVIGATION

- Surveying for Schools and Scouts*: Richardson (Philip).
Surveying for Everyone: William (Sheldon Press).
A Little Book on Map Projection: Garnett (Philip).
Mathematical Geography (2 vols.): Jameson and Ormsby (Pitman).
Map Work: Bryant and Hughes (Oxford University Press).
Map Projections: Hinks (Cambridge University Press).
Maps and Survey: Hinks (Cambridge University Press).
The Study of Map Projections: Steers (University of London Press).
Experimental Geography: Dingwall (Bell).
Survey and Field Astronomy: Reeves (Royal Geog. Society).

Mapwork and Practical Geography: Bygott (University Tutorial Press).
The Nautical Almanack (H.M.S.O.).

ASTRONOMY, TIME, TIDES

Through My Telescope: Hay (Murray).
Worlds Without End: Spencer Jones (English University Press).
The Children's Book of the Heavens: Proctor (Harrap).
Starland: Bell (Cassell).
Astronomy: Hinks (Home University Library).
The Wonders of Modern Astronomy: Macpherson (Seeley).
The Romance of Modern Astronomy: Macpherson (Seeley).
The Romance of the Planets: Proctor (Harpers).
Stars and How to Identify Them: Maunder (Epworth).
The Stars and Their Mysteries: Gibson (Seeley).
Sidelights of Astronomy: Newcomb (Harpers).
The Great Ball on which we Live: Gibson (Seeley).
Makers of Astronomy: Macpherson (Oxford).
A Key to the Stars: Woolley (Blackie).
Astronomy: Dyson (Dent).
Stars and Atoms: Eddington (Cambridge University Press).
The Story of the Heavens: Ball (Cassell).
The Universe Around Us: Jeans (Cambridge).
Through Space and Time: Jeans (Cambridge).
Time Measurement: Bolton (Bell).
The Tides: Darwin (Murray).

GENERAL WORKS

The Post Office Guide, Whitaker's Almanack, Bradshaw.
Modelling in Paper and Cardboard: Locke and Robinson (Macmillan).
Ruler and Compass: Hudson (Longmans).
Easy Mathematics: Lodge (Macmillan).
Graph Book: Durell and Siddons (Bell).
First Ideas on the Calculus: Crosland (Ginn).
Calculus Made Easy: Thompson (Macmillan).
Exponentials Made Easy: Bray (Macmillan).
Flatland: Abbott (Blackwell).

- Readable Relativity*: Durell (Bell).
Housecraft Arithmetic: Mellor and Pearson (Longmans).
The Small Investor: Parkinson (Blackie).
Groundwork of Book-keeping: Gordon (Bell).
The Use of Graphs in Commerce and Industry: Palmer (Bell).
Finance (2 vols.): Palmer (Bell).

PRACTICAL MATHEMATICS, MACHINES, SCIENCE

- Readable School Mechanics*: Fawdry (Bell).
How it Works: Williams (Nelson).
Romance of Modern Engineering: Williams (Seeley).
The Wonders of Mechanical Ingenuity: Williams (Seeley).
Engines: Andrade (Bell).
Spinning Tops: Perry (S.P.C.K.).
Living Machinery: Hill (Bell).
Engines of the Human Body: Keith (Williams and Nargate).
The Mechanism of Nature: Andrade (Bell).
Force and the Study of Energy: Wood (Bell).
The Mysterious Universe: Jeans ('Pelican' Series).
The Nature of the Physical World: Eddington (Dent).
Slide-Rule Notes: Dunlop and Jackson (Longmans).
Modern Instruments of Calculation (ed. Horsburgh) (Bell).
(Also contains an account of the life and work of Napier).

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